Conceptual and historical aspects of von Neumann's no hidden variables proof.

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∞ Motivation ∞

- I came across Grete Hermann's 1935 **critique** of von Neumann's argument only to find it **very much ignored**.

- Reading von Neumann's book I could not see why one of his assumptions is supposed to be **'silly'**.

  This judgement appears to be **totally misattributed**.

- In order to give a **better judgement** of the argument I want to know what the argument actually tells us.

∞ Outline ∞

1. **Von Neumann's no hidden variable argument**

2. **It's supposed sillyness and the standard view**

3. **A new look:**
   - The argument is not silly but merely unconvincing

4. **Grete Hermann's critique on von Neumanns' argument**
   - Similar critique as John Bell gave thirty years later.
   - The reception of her work

5. **What actually does the von Neumann argument tell us?**

∞ Von Neumann's 1932 no hidden variable argument ∞

Von Neumann: What reasons can be given for the **dispersion** found in some quantum ensembles?

- **(Case I)**: The individual systems differ in **additional parameters**, which are not known to us, whose values determine precise outcomes of measurements.
  $\Longrightarrow$ **deterministic hidden variables**.
- **(Case II)**: 'All individual systems are in the **same state**, but the laws of nature are **not causal**'.

ad Case I: No **physical method** exists of dividing a dispersive ensemble into dispersion free ensembles, because of the unavoidable **measurement disturbance**.

• However, it nevertheless is possible **conceptually** to think of each ensemble as composed out of two (or more) **dispersion free** subensembles.

• Von Neumann's proof is intended to show that even the latter is **impossible**.

- Von Neumann's notion of a **hidden variables** theory :

It is a causal theory which defines the state of the system 'absolutely' by supplying 'additional numerical data'– this additional data are the 'hidden parameters'.

'If we were to know all of these, then we could give the values of all physical quantities exactly and with certainty.'

A **hidden variables theory** now is one which is ' *[...] in agreement with experiment, and which gives the statistical assertions of quantum mechanics when only ϕ is given (and an averaging is performed over the other coordinates).'*

- **Mathematical characterization of hidden variables**:

Every physically realizable state can be represented as a mixture of **homogeneous dispersion-free states**:

(α') An ensemble is **dispersion free** if

$$Exp(\mathfrak{R}^2) = (Exp(\mathfrak{R}))^2 \quad , \forall\, \mathfrak{R}.$$

(β') An ensemble is **homogeneous** or **pure** if the statistics of it is the same as that of **any** of its subensembles.

$$E = a\, E_1 + bE_2 \implies E = E_1 = E_2.$$

∞ The question of 'hidden parameters' ∞

- Now **suppose** there are homogeneous ensembles, then **if** hidden variables exist (any dispersive ensemble can thus be split into two or more non-dispersive ones), the homogeneous ensembles **must be** dispersion free:

$\Longrightarrow$ **No dispersive ensemble can be homogeneous.**

This is what according to von Neumann is implied by the existence of hidden variables.

- In **classical** Kolmogorov type statistical ensembles all and only dispersion free ensembles are homogeneous.

What about quantum mechanical ensembles?

Von Neumann proofs that this is not the case: all homogeneous ensembles are dispersive.

The Proof.

We have to consider a theory **general enough** to deal with both Case I and Case II statistical theories.

Von Neumann implements this as follows. Every physical **ensemble** determines a **functional** Exp, which is supposed to characterize it completely from a statistical point of view.

The Exp-functional must satisfy the following general **assumptions**:

- **(0)** To each observable of a quantum mechanical system **corresponds** a unique hypermaximal Hermitian **operator** in Hilbert space. This correspondence is **one-to-one**.
- **(I)** If the observable $\mathfrak{R}$ has operator R then the observable $f(\mathfrak{R})$ has the operator $f(R)$.
- **(II)** If the observables $\mathfrak{R}$, $\mathfrak{S}$, ... have the operators R, S, ..., then the observable $\mathfrak{R} + \mathfrak{S} + \ldots$ has the operator $R + S + \ldots$. (the **simultaneous measurability** of $\mathfrak{R} + \mathfrak{S} + \ldots$ is **not** assumed.)
- **(A')** If the observable $\mathfrak{R}$ is by nature a **nonnegative** quantity then $Exp(\mathfrak{R}) \geq 0$.

(B') If $\mathfrak{R} + \mathfrak{S} + \ldots$ are arbitrary observables and $a,\ b,\ \ldots$ real numbers, then $Exp(a\,\mathfrak{R} + b\,\mathfrak{S} + \ldots) = a\,Exp(\mathfrak{R}) + b\,Exp(\mathfrak{S}) + \ldots$.

• Von Neumann demonstrated on the basis of **all** these assumptions that there exists a **linear, semidefinite Hermitian matrix** U_{mn} such that for any observable $\mathfrak{R}$

$$Exp(\mathfrak{R}) = \sum U_{nm} R_{mn} \; = \; \mathrm{Tr}\,(\mathrm{UR}). \tag{1}$$

• Thus every ensemble in quantum mechanics is characterised by a **statistical operator** known as the **density operator** (or **density matrix**).

(Note: Gleason (1957) proofs this also, but requires **(B')** only for **commuting** observables, $\mathrm{Dim}(\mathcal{H}) \geq 3$.)

• What are (I) the **dispersion free** and (II) the **homogeneous ensembles** among the density operators U?

I: What U have $\mathrm{Tr}(\mathrm{UR}^2) = [\mathrm{Tr}(\mathrm{UR})]^2$ for all R?

$\Longrightarrow$ **No** U fulfill this requirement, thus **no dispersion free states exists**.

II: He next proofs that homogeneous ensembles **do exist**:

$\Longrightarrow$ The homogeneous ensembles are the **pure** quantum states (**one-dimensional** projection operators).

I & **II** $\Longrightarrow$ **All** ensembles show **dispersion**, even the homogeneous ones.

∞ Back to the question of 'hidden parameters' ∞

Can the **dispersion** in the homogeneous ensembles be explained by the fact that the states are mixtures of several states, '*which together would determine everything causally, i.e., lead to dispersion free ensembles?*'

Von Neumann:
' The statistics of the homogeneous [dispersive] ensembles would then have resulted from from the averaging over all actual states of which it was composed [...]. But this is impossible for two reasons:

First, because then the homogeneous ensembles in question could be represented as a mixture of two different ensembles, contrary to its definition.

Second, because the dispersion free ensembles [. . .] do not exist.'

$\Longrightarrow$ Homogeneous ensembles exist that are not dispersion free, therefore the assumption of the existence of hidden variables is refuted.

John Bell paves the way for the standard view

In 1964 (published 1966) John Bell intended to show what the problem with von Neumann's argument was, after he 'saw the impossible done'. He tracked it down to the assumption **(B')**:

$$Exp(a\,\mathfrak{R} + b\,\mathfrak{S}) = a\,Exp(\mathfrak{R}) + b\,Exp(\mathfrak{S}). \qquad (2)$$

This relation holds true for quantum mechanics, **irrespective** of whether the operators R and S commute.

- **Bell reasons**: It is required by von Neumann of the hypothetical dispersion free states **also**. But for a dispersion free state the expectation value must equal one of the operator's **eigenvalues**. But eigenvalues **do not** generally combine linearly.

Example: $(\sigma_x + \sigma_y)$ with eigenvalues $\pm\sqrt{2}$
$\sigma_x, \ \sigma_y$, with eigenvalues ± 1.

In some cases the **addivity of expectation values** gives the requirement for **additivity of eigenvalues.** The latter is generally not true.

John Bell:

‘The essential assumption can be criticized as follows. [. . .] A measurement of a sum of noncommuting observables cannot be made by combining trivially the results of separable observations on the two terms – it requires a quite distinct experiment. [. . .] But this explanation of the nonadditivity of allowed values also established the nontriviality of the additivity of expectation values. The latter is quite a peculiar property of quantum mechanical states, not to be expected *a priori*.’ (Bell, 1964 (1966))

‘. . . for the individual results are eigenvalues and eigenvalues of linearly related operators are not linearly related. [. . .] His very general and plausible postulate is absurd.’ (Bell, 1982)

The once superior proof becomes allegedly silly.

’A third of a century passed before John Bell, 1966, rediscovered the fact that von Neumann’s no-hidden-variables proof was based on an assumption that can only be described as silly – so silly, in fact, that one is led to wonder whether the proof was ever studied by either the students or those who appealed to it to rescue them from speculative adventures.’
(N. David Mermin, 1993)

'Yet the von Neumann proof, if you actually come to grips with it, falls apart in your hands! There is *nothing* to it. It's not just flawed, it's *silly*! ...When you translate [his assumptions] into terms of physical disposition they're nonsense. You may quote me on that: The proof of von Neumann is not merely false but *foolish*!' (John Bell, 1988, as cited by N. David Mermin.)

The Standard View

N. David Mermin considers the assumption:

$$v(A + B) = v(A) + v(B) \tag{3}$$

'Von Neumann's silly assumption was to impose the condition on a hidden variables theory even when A and B do not commute.' (N. David Mermin, 1993)

- Almost all discussions of von Neumann's argument use this **value addition rule** as an assumption he is supposed to have made.

The Kochen-Specker theorem (1967) is **indeed** phrased in terms of values. They require $v(f(A)) = f(v(A))$ from which the additivity rule for values for commuting A and B follows.

However, von Neumann did **explicitly not use** the additivity rule for values of noncommuting observables as an assumption.

'In general we call two (or more) quantities $\mathfrak{R}$, $\mathfrak{S}$ simultaneously measurable if there is an arrangement which measures both simultaneously in the same system [...] Fur such quantities, and a function $f(x, y)$ of two variables, we can also define the quantity $f(\mathfrak{R}, \mathfrak{S})$. This is measured if we measure $\mathfrak{R}$, $\mathfrak{S}$ simultaneously – if the values a, b are found for these, then the value of $f(\mathfrak{R}, \mathfrak{S})$ is $f(a, b)$. But it should be realised that it is completely meaningless to try form $f(\mathfrak{R}, \mathfrak{S})$ if $\mathfrak{R}$, $\mathfrak{S}$ are not simultaneously measurable: there is no way of giving the corresponding measuring arrangement.

However, the investigation of the physical quantities related to a single object $\mathcal{S}$ is not the only thing which can be done – especially if doubt exist relative to the simultaneous measurability of several quantities. In such cases it is also possible to observe great statistical ensembles which consist of many systems. In such an ensemble we do not measure the "value" of a quantity $\mathfrak{R}$ but its distribution of values. [...] Even if two (or more) quantities $\mathfrak{R}$, $\mathfrak{S}$ in a single system $\mathcal{S}$ are not simultaneously measurable, their probability distributions in a given ensemble can be obtained with arbitrary accuracy if N is sufficiently large.'

(J. von Neumann, page 298, 1955 English translation)

∞ The argument is not silly yet unconvincing ∞

• **The argument is not silly**. Von Neumann **nowhere** uses additivity of eigenvalues as an assumption. It is a result of the additivity of expectation values and the fact that for dispersion free states (the ones he considers) expectation values happen to be equal to the **numerical value** of the eigenvalues.

What did Bell show us?

'Bell clarified the situation by pointing out that it was not the "objective verified predictions of quantum mechanics", but rather his arbitrary additivity assumption, postulated to be also valid for dispersion free states, that precluded the possibility of hidden variables.' (Max Jammer, 1974)

This is **wrong**. The assumption doesn't preclude it, it shows that the argument **trivially** proofs that the hidden variables subject to his assumptions are not possible.

• It shows that the proof is **trivial** in the case all the dispersion free states are required to be of the form U (a density operator) . Thus there exist no dispersion free quantum states.

The argument is unconvincing .

- The additivity rule for expectation values in the case of incompatible observables cannot be justified in the light of the **Bohrian point** that **contexts of measurement** play a role in defining the nature of quantum reality.

'There is no reason to demand it individually of the hypothetical dispersion free states ...' (Bell, 1964). And thus there is no reason to demand that the dispersion free states are of the form of a density operator U.

This is ironic in two senses:

1) It is a sort of **judo-like manouvre** (Abner Shimony, 1984):

A Bohrian consideration **saves** hidden variables against von Neumann. '[Bell] cited Bohr in order to vindicate a family of hidden variables theories ...' (Jammer, 1974):

'[The assumption] is seen to be quite unreasonable when one remembers with Bohr "the impossibility of any sharp distinction between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which phenomena appear."' (Bell, 1964)

2) The additivity rule, although it has no justification, nevertheless does hold **true** in quantum mechanics. It is thus **surprising** that it holds true.

A priori no statistical results can be expected between $Exp(\mathfrak{R})$ and $Exp(\mathfrak{S})$. The additivity holds because '**it so happens** that the other axioms and postulates of quantum theory conspire to make $Exp(\mathfrak{R})$ expressible at $\int \psi^* \, R \, \psi \, dx$.' (Belinfante, 1973).

That the additivity rule holds is because of the accidental fact that $Exp(\mathfrak{R}) \;=\; \langle\, \phi |\, R \,|\, \phi \rangle$ in quantum mechanics and therefore that (pure case):

$$\langle\, \phi |\, (R + S) \,|\, \phi \rangle \;=\; \langle\, \phi |\, R \,|\, \phi \rangle \;+\; \langle\, \phi |\, S \,|\, \phi \rangle \,. \tag{4}$$

Conclusion: It doesn't count against the theorem that the nonexistence of dispersion free states is so easy to prove in the case of the hidden variables he considers (A lot easier than von Neumann thought). The **real issue** is his notion of hidden variable and the plausibility of the premisses in the proof.

∞ Grete Hermann's critique on von Neumann's argument ∞

Grete Hermann published in 1935 a treatise called 'Die Naturphilosophischen Grundlagen der Quantenmechanik', published in the *Abhandlungen der Fries'schen Schule.*

- Section 7: **'The circularity of von Neumann's proof.'**, page 99:

$$Exp(\mathfrak{R} + \mathfrak{S}) = Exp(\mathfrak{R}) + Exp(\mathfrak{S}) \tag{5}$$

'With this assumption the proof of von Neumann either succeeds or fails.'

Grete Hermann gives rise to **all** that John Bell had concluded (but with a different argument):

1) Hermann concluded, just like Bell, that von Neumann **precluded** the non-existence of dispersion free states because the additivity rule is too restrictive.

2) She commented upon the **problematic status** of the additivity rule in the light of the impossibility of simultaneous measurement of noncommuting observables.

ad 2): 'Not trivial however is the relation for quantum mechanical quantities for which indeterminacy relations hold. In fact, the sum of two such quantities is not even defined: Because sharp measurement of one of them excludes sharp measurement of the other one and thus both quantities cannot have sharp values at the same time, the commonly used definition of the sum of two quantities brakes down.

Thus, for the above determined notion of a sum of two, not jointly measurable quantities, the above mentioned equation requires a proof. [...]Von Neumann concludes that for ensembles of systems with identical wave functions, and also for all ensembles, the sum rule for expectation values holds, even for such quantities that cannot be measured simultaneously.'
(Grete Hermann, page 100, 1935 (my translation))

ad 1): To interpret the pure case $|\phi\rangle$ ensembles to be the dispersion free states for which $Exp(\mathfrak{R}) = \langle\phi|R|\phi\rangle$ must hold, is to restrict further specification of the state by hidden variables. '*The impossibility of such a specification is just the thesis to be proved.*' She regards the proof **circular**.

The right conclusion is not that this precludes hidden variables (just like Bell did), but that the proof holds only for **a limited class of hidden variables**, namely those that obey **(B')**.

Grete Hermann, 1901-1984.

∞ The reception of her work ∞

With hindsight we can say that Grete Hermann was **ahead of her time**. Only really after John Bell (1964) the limited applicability of von Neumann's proof becomes known.

Heisenberg and von Weizsäcker surely knew of her criticism.

Why was her criticism ignored at the time?

1. Von Neumann's proof was sort of **holy**:

'The truth, however, happens to be that for decades nobody spoke up against von Neumann's arguments, and that his conclusions were quoted by some as the gospel.' (F. J. Belinfante, 1973)

'Now the mere mention of concealed variables is sufficient to automatically elicit from the elect the remark that John von Neumann gave absolute proof that this way out is not possible. To me it is a curious spectacle to see the unanimity with wich the members of a certain circle accept the rigor of von Neumann's proof' (Bridgman, 1960)

‘He [Bohr] came for a public lecture.... At the end of the lecture he left and the discussion proceeded without him. Some speakers attacked his qualitative arguments –there seems lots of loopholes. The Bohrians did not clarify the arguments; they mentioned the alleged proof by von Neumann and that settled the matter.[...]. Yet, like magic, the mere name of ”von Neumann” and the mere word ”proof” silenced the objectors.’ (Feyerabend)

2. No english translation was available of von Neumann’s book untill 1955.

3. Grete Herman published her treatise in a not well known series of books. The summary that appeared in 1935 in the good and well-read journal ’Die Naturwissenschaften’ did not contain the argument against von Neumann, but only her Kantian ideas.

Why is her criticism still not known?

1. Her treatise is still not translated into English.

2. Max Jammer doesn’t mention her in connection to Bell’s criticism on von Neumann’s argument.

3. People who mention Grete Hermann's criticism only mention what Jammer wrote. One exception is Lena Soler, but she wrote about her in French.

4. Von Neumann's argument itself is not widely studied:

'A book more frequently referred to than read by physicists because of its mathematical sophistication.' (Redhead, 1987)

'Well, I suppose that they regard von Neumann's book as a perfectly adequate formal treatment for pedants, people who like that sort of thing [formal mathematics]. They wouldn't read it themselves but they're glad somebody has done all that hard work!' (Bastin, 1977)

5. James Albertson (1961) in his accessible and therefore well-studied Dirac-formulation of the proof is not critical at all and relegates all assumptions, including the problematic one, to an appendix.

∞ What actually does the argument tell us ∞

Von Neumann proofs that a certain limited class of hidden variables is excluded. Those that obey his assumptions. He himself thought he was completely general and unrestrictive.

'Nevertheless, under all circumstances, $Exp\,(\mathfrak{R} + \mathfrak{S}) \;=\; Exp\,(\mathfrak{R}) + Exp\,(\mathfrak{S})$.' (von Neumann, 1955)

• Why? Because it holds true in quantum mechanics:
'In each state ϕ the expectation values behave additively: $(R\phi, \phi) + (S\phi, \phi) = ((R + S)\phi, \phi)$. The same holds for several summands. We now incorporate this fact into our general set-up (at this point not yet specialized to quantum mechanics).' (von Neumann, 1955)

• Von Neumann assumes that all $Exp(\mathfrak{R})$ for hidden variables must arise from the assumptions he has made. This can be contested since **(B')** is very much questionable. It is unreasonable to require it of the dispersion free states as represented by measures on a classical probability space. It can be required only for those probability measures corresponding to the statistical states of quantum mechanics.

- Von Neumann **does not** consider averaging over the dispersion free states to reproduce quantum mechanics, contrary to what he claims he does.

We can get the homogeneous ensembles with dispersion that quantum mechanics predicts by using dispersion free hidden variables as a classical probability measure **and averaging over them**.

$$Exp(R)_U \;=\; \mathrm{Tr}\,[UR] \;=\; \int R[\lambda]\,\rho_U(\lambda)\,d\lambda \tag{6}$$

On a two-dimensional Hilbert space Bell gave an example of this.

What does the proof tell us?

(I): It excludes hidden variables where **each** hidden variable state must reproduce quantum mechanics. Quantum states cannot be represented as one-point measures on the hypothetical phase space. **Necessarily we need a richer theory**. Thus only when averaged over, can the hidden variables reproduce quantum mechanics.

- **The richer theory.** To a particular quantum state U **more** hidden variable states correspond, a distribution $\rho(\lambda)$. Of course the ρ's have to give the same expectation values as U does for all quantum mechanical observables. But it can even be so that the hidden variables theory has **new** physical quantities (such as λ itself), that have different expectation values in the different ρ's that correspond to the single U.

(II): The issue is not that we require additivity for eigenvalues (the so-called 'silly' assumption), but that because von Neumann's conception of hidden variables is too restricted, that $(\mathbf{B}')$ becomes required of dispersion free hidden variable states and thus phrased in terms of eigenvalues.

Happy End

Max Jammer comments on the fact that the proof, although ‘did not demonstrate that quantum mechanical ensembles cannot be decomposed into *any* kind of dispersion-free states’, can nevertheless not be dismissed as ’nugatory’. ‘True, in view of von Neumann’s excessively restrictive assumptions it is not an *impossibility proof* of any conceivable class of hidden variables, but it is a *completeness proof* since this formalism with the inclusion of the additivity postulate does not admit non-quantum mechanical ensembles. It may even be regarded a *consistency proof* of this formalism with its usual interpretation.’ (Jammer, 1974)