An invariant variation principle for the motion of many electrical mass particles.

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With 2 illustrations. (Submitted on 1st October 1929.)

[Translation[1] by M.P. Seevinck of '*Ein invarianter Variationssatz für die Bewegung mehrerer elektrischer Massenteilchen.*' by A.D. Fokker, *Zeitschrift für Physik*, vol. 58, 386 – 393 (1929).]

A variation principle for point mechanics of many electrical mass particles is put forward which is invariant under Lorentz transformations and in which, through the use of retarded and advancd potentials, the motions of the particles are included in a fully symmetrical way. The form of the conservation laws for energy and momentum is determined and the consequences for the definition of these quantitities is discussed.

Quantum mechanics has, in accordance to the correspondence principle, taken its departure in the classical theory of the dynamics of a single particle, and has formulated its laws and methods for the single-particle problem in accordance with the canonical Hamiltonian equations. However, when considering the interactions between many particles, it [quantum mechanics] has not found a form which satisfies the requirement of invariance under Lorentz-transformations. Here the already existing work precisely hits the bottom of the classical theory.

Because of this reason it could be useful to construct and further develop[2] an attempt to obtain the foundations of such a theory in the form of a Hamilton-like variation principle, which would concern itself exclusively with the motions of particles and their interactions on eachother and which could completely do without any consideration of a field[3].

When one wants to rewrite the usual version of the variation principle for a mass particle

$$\delta \int (T - U)\, dt = 0,$$

[1] *This work was completed during a visit at the Center for Time, University of Sydney, Australia. Their hospitality and support is kindly acknowledged.*

[2] Cf. Physica **9**, 33, 1929.

[3] W.Heisenberg and W. Pauli have recently set out to consider the field. Zeitschrift für Physik **56**, 1, 1929.

relativistically, then one should, as is generally known, write in place of Tdt:

$$-mc^2(1-\frac{v^2}{c^2})^{1/2}dt = -mcds,$$

where m is the mass, v the velocity of the particle and c the speed of light.

In order to extend the expression Udt to an invariant scalar, we remind ourselves that the potential energy is only the time component of a covariant four-vector, whose space components are represented through the negative components $-A_x$, $-A_y$, $-A_z$ of some momentum, which, in analogy to the energy component, may be called a 'potential" momentum. Such an extension of the four-vector allows us to write instead of $-Udt$ the scalar:

$$-Udt + A_x d_x + A_y d_y + A_z d_z.$$

In the case of the motion of an electrical particle, the covariant potential energy-momentum vector is given by the product of its charge and field potential, whereby the vector potential can be divided by $-c$.

The potentials at a receiving point X, with space-time coordinates

$$x^0, x^1, x^2, x^3 \ (=) \ t, x, y, z,$$

can originate from a charge e, whose motion is given in case its time and space coordinates w^i $(i = 0, 1, 2, 3)$ can be represented as a function of a parameter u. In ordr to find the potentials, we must, following Liénard and Wiechert, determine the "effective" space-time point W of the generating charge from which one can reach the receiving point with the speed of light. This is thus given by

$$c^2(x^0-w^0)^2 - (x^1-w^1)^2 - (x^2-w^2)^2 - (x^3-w^3)^2 = R^2 = 0,$$

and thus

$$x^0 - w^0 = \frac{r}{c}, \quad r = \sqrt{(x^1-w^1)^2 + (x^2-w^2)^2 + (x^3-w^3)^2}.$$

If we denote by v_r the radial velocity components of the generating charge which is directed towards x^i at this space-time point, then the potentials ϕ and a_x, a_y, a_z are given by:

$$\Phi = \frac{e}{4\pi}\frac{1}{r\,(1-\frac{v_r}{c})}, \quad a_x = \frac{e}{4\pi c}\frac{\frac{dw_1}{dw^0}}{r\,(1-\frac{v_r}{c})}, \ \text{etc.}$$

In order to construct covariant four-vectors out of this, we write

$$\Phi = \frac{e}{4\pi c} \frac{c^2 dw^0}{c^2(x^0 - w^0)\,dw^0 - (x^1 - w^1)\,dw^1 \cdots} = \frac{e}{4\pi c} \frac{dw_0}{(R \cdot dw)},$$

and correspondingly

$$-\frac{a_x}{c} = \frac{e}{4\pi c} \frac{-dw^1}{c^2(x^0 - w^0)\,dw^0 - (x^1 - w^1)\,dw^1 \cdots} = \frac{e}{4\pi c} \frac{dw_1}{(R \cdot dw)}, \quad \text{etc.}$$

Here $(R \cdot dw)$ denotes the product of the connecting line (or connecting radius)

$$R^i = x^i - w^i, \quad (i = 0, 1, 2, 3)$$

and the four-dimensional element of motion dw^i.

This gives us the retarded potentials generated by e in X from W.

Consider now the motions of charges $e_2,\ e_3,\ \ldots$ which are given by their space and time coordinates $y^i(u),\ z^i(v),\ \ldots$, as functions of the parameter $u,\ v,\ \ldots$. If one wants to determine the motion x^i of the mass particle m_1, with charge e_1, under the influence of these [charges $e_2,\ e_3,\ \ldots$], then, according to what has been said, one should formulate the following variatonal law

$$0 = \delta \Big[\int -m_1 c\, ds_1 - \frac{e_1 e_2}{4\pi c} \int \frac{(dx \cdot dy)}{(R \cdot dy)} - \frac{e_1 e_2}{4\pi c} \int \frac{(dx \cdot dz)}{(S \cdot dz)} - \cdots \Big],$$

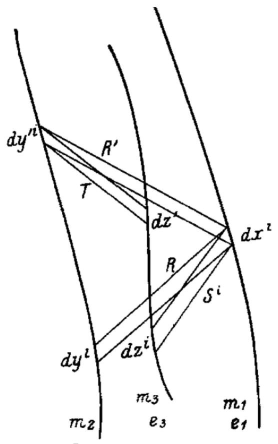

Figure 1

where ds_1 is the value of the curve element dx^i, and $R^i, S^i, \ldots$ are the connecting lines that connect the line element dx^i with the corresponding line elements dy^i $dz^i, \ldots$, and which have zero length (Fig 1.)

We want to adhere to the following: to a particular division of the line element dx^i of the motion of the charge e_1 we can associate an infinitesimal division of the motions of the other charges, in such a way that

$$R^2 = 0, \ \ S^2 = 0, \ \ \text{etc},$$

holds all the time. A consequence hereof is that with this correlation it is always the case that

$$(R \cdot dy) = (R \cdot dx), \quad (S \cdot dz) = (S \cdot dx), \ \ \text{etc.}$$

If the motion of the charges $e_1, e_3, \ldots$ is given to us, and we would like to determine the motion of the charge e_2 that has mass m_2, then we would let the line element dy'^i of this motion (with curve length ds_2) correspond to line elements dx^i, dz'^i, ... of the other motions of the charges via connecting lines R', T, ..., and we would construct the law

$$0 = \delta \Big[\int -m_2 c\, ds_2 - \frac{e_1 e_2}{4\pi c} \int \frac{(dy' \cdot dx)}{(R' \cdot dx)} - \frac{e_2 e_3}{4\pi c} \int \frac{(dy' \cdot dz')}{(T \cdot dz)} - \cdots \Big].$$

The same should be performed for the third charge and also for the other charges.

Instead of each particle having its own variational law, it is desirable to be able to grasp the total system through one interaction principle. That, however, is opposed to the fact that the action integral

$$\frac{e_1 e_2}{4\pi c} \int \frac{(dx \cdot dy)}{(R \cdot dy)},$$

although admittingly taking into account the influence on e_1 by the retarded action of e_2, does not agree with the corresponding integral that denotes the reciprocal influence on the motion of e_2 by the retarded action of e_1. We can however remark that because of $(R \cdot dy) = (R \cdot dx)$ the denoted integral in the form

$$\frac{e_2 e_1}{4\pi c} \int \frac{(dy \cdot dx)}{(R \cdot dx)},$$

would represent the action on e_2 by the advanced potentials that are generated by e_1.

One can perfectly well regard the habit to calculate only with retarded integrals to be arbitrary —which is today not stressed for the first time. In view of a too greatly obtained reciprocity of the interaction, it is appropriate to formulate the exerted action by e_2 on e_1 as half retarded and half advanced. That would have the consequence that in the variational law of the motion of the first particle the action of the second particle will be represented by the very same integral as is the case for how the action of the first particle is represented in the law of motion of the second particle. That further allows one to formulate a single variational law for the joint motion, and which is

$$0 = \delta \Big[\sum \int -m_i c\, ds_i - \sum \frac{e_i e_j}{8\pi c} \Big\{ \int \frac{(dx \cdot dy)}{(R \cdot dy)} + \int \frac{(dy' \cdot dx)}{(R' \cdot dx)} \Big\} \Big],$$

where the first summation must be performed over the worldlines that correspond to the single particles, and the second summation over the interaction integrals corresponding to particle pairs.

This law is completely invariant under Lorentz transformations. There is not a single reference to any field whatsoever. The symmetry and reciprocity in the interactions of the particles is optimal. We have to admit that it is more concerned with the system of motions than with the system of particles.

The notion of a system of particles would require a certain assignment of its elements of motion. It is the case that this assignment can be defined uniquely, but not invariantly, or invariantly but not uniquely; it is however impossible to define the system of particles both invariantly and uniquely at the same time. It is therefore unavoidable to regard the appearances as a system of motions and not as a moving system of particles.

We would now like to carry out a variation of the motion of charge e_1 by displacing each space-time point x^i of this motion by an infinitesimal space-time line element δx^i, and then calculate the variation of the interaction integral

$$\frac{e_1 e_2}{8\pi c}\,\delta\int\frac{(dx\cdot dy)}{(R\cdot dy)},$$

which corresponds to the retarded action of e_2 on e_1. During the variation the correspondence between the elements dx^i and dy^i must be maintained: The connecting line connecting line R^i between them should represent a light ray ($R = 0$), and to achieve this the variation δx^i should itself be accompanied by the displacement

$$\delta y^i = \frac{dy^i}{(R\cdot dy)}(R\cdot\delta x)$$

of the motion of the charge e_2. This motion is therefore not changed at all, only the necessary correspondence in the interaction integral between the elements of motion of the particles is ensured through $(R\cdot\delta y) = (R\cdot\delta x)$ and thus through $\delta R^2 = 0$. Also, it remains the case that for the corresponding elements $(R\cdot dy) = (R\cdot dx)$ holds, because of which we can use for the variation of the denominator the variation of $(R\cdot dx)$.

We can therefore write:

$$\delta\,\frac{(dx\cdot dy)}{(R\cdot dy)} = \frac{(\delta x\cdot dy)+(dx\cdot\delta y)}{(R\cdot dy)} - \frac{(dx\cdot dy)\delta(R\cdot dx)}{(R\cdot dy)(R\cdot dx)}.$$

This expression can be partially integrated while requiring that the variation should vanish at the integration boundaries. In integrating one obtains when taking the value of δ^i into consideration:

$$\sum_{i,m}\delta x^i\Big[-d\Big\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)}\Big\} - R_i\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dx)} - dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} + R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2}\Big].$$

the expression in the curly brackets (braces) gives the increase in kinetic energy and momentum (for $i = 0$, respectively $i = 1, 2, 3$), in short, the exerted force on the first particle. It can be seen that, except for the usual neglected dependencies on the velocities in the derivations, the force in part depends on the acceleration of the exerting charge, and in part it does not depend on it. This corresponds to the electrostatic action (last term) and the so-called radiative action (term with curly brackets and the next).

In our formulation we also encounter the advanced action of e_2 in the integral

$$-\frac{e_1 e_2}{8\pi c}\int \frac{(dx\cdot dy')}{(R'\cdot dx)} = -\frac{e_1 e_2}{8\pi c}\int \frac{(dx\cdot dy')}{(R'\cdot dy')}.$$

In case we deal with the variation in the same manner as before, and taking into account that here $\delta R'^i = \delta'^i - \delta x^i$, we can easily write down the requirement for the vanashing of variation of the total integral:

$$\begin{aligned}0 =& d\left(m_1 c\frac{dx_i}{ds_1}\right) + \frac{e_1 e_2}{8\pi c}\Big[d\Big\{\frac{dy_i}{(R\cdot dy)} - R_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)}\Big\} + d\Big\{\frac{dy'_i}{(R'\cdot dy')} - R'_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')}\Big\}\\ &+R_i\sum\frac{dy^m}{(R\cdot dy)}d\frac{dx_m}{(R\cdot dx)} + dx_i\frac{(dx\cdot dy)}{(R\cdot dx)(R\cdot dy)} - R_i\frac{(dx\cdot dy)^2}{(R\cdot dx)(R\cdot dy)^2}\\ &+R'_i\sum\frac{dy'^m}{(R'\cdot dy')}d\frac{dx_m}{(R'\cdot dx)} - dx_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')} + R'_i\frac{(dx\cdot dy')^2}{(R'\cdot dx)(R'\cdot dy')^2}\Big].\end{aligned}$$

In the expression inside the large brackets, which, when taken to the left-hand side with a minus sign, signifies the covariant vector indicating the increase of kinetic energy and momentum from e_2 to e_1 (for $i = 0$, resp. $i = 1, 2, 3$), we can find some total differentials.

These can be interpreted as the potential energy and momentum of charge e_1 as caused by the presence and motion of e_2.

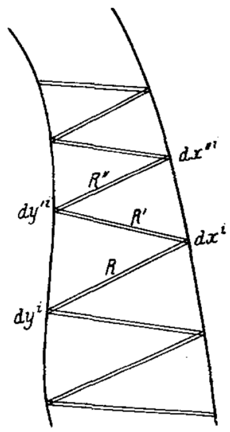

Figure 2

A similarly built equation holds for the motion of the second particle under the influence of the charge e_1, in which two elements of the motion of e_1 enter. Writing down this equation for the element dy'^i (Fig 2.), which is connected to the previously considered element dx^i via the correlation $R' = 0$, and via the correlation $R'' = 0$ to a second element dx'', we obtain

$$
\begin{aligned}
0 =& d\left(m_1 c\frac{dy'_i}{ds_2}\right) + \frac{e_1e_2}{8\pi c}\Big[d\Big\{\frac{dx_i}{(R'\cdot dx)} - R_i\frac{(dy'\cdot dx)}{(R'\cdot dx)(R'\cdot dy')}\Big\} + d\Big\{\frac{dx''_i}{(R''\cdot dx'')} - R''_i\frac{(dy'\cdot dx'')}{(R''\cdot dx'')(R''\cdot dy')}\Big\}\\
&+R'_i\sum\frac{dx^m}{(R'\cdot dx)}d\frac{dy'_m}{(R'\cdot dy')} + dy'^i\frac{(dy'\cdot dx)}{(R'\cdot dy')(R'\cdot dx)} - R'_i\frac{(dx\cdot dy')^2}{(R'\cdot dy')(R'\cdot dx)^2}\\
&+R''_i\sum\frac{dx''^m}{(R''\cdot dx'')}d\frac{dy'_m}{(R''\cdot dy')} - dy'_i\frac{(dy'\cdot dx'')}{(R''\cdot dy')(R''\cdot dx'')} + R''_i\frac{(dy'\cdot dx'')^2}{(R''\cdot dy')(R''\cdot dx'')^2}\Big].
\end{aligned}
$$

It is now important to remark that the third row in this expression and the last row of the previous one together form a total differential. Both indicate the reciprocal action which takes place along the contact radius R' between the elements of motion dx_i and dy'_i. Together they give:

$$
\frac{e_1e_2}{8\pi c}d\left\{R'_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')}\right\}.
$$

This shows us the way towards the form in which the conservation laws for energy and momentum ($i = 0, 1, 2, 3$) apply here. For simplicity we will look at the interaction between only two particles.

We must draw a continually back and forth going zigzag-chain (Fig. 2), and we must, for all elements of motion that are coordinated on two such neighbouring chains, write down the equations of motion and sum everything together.

In this summation every turning point of the zigzag-chain, when it lies in the motion of e_1 or e_2, gives a (kinetic) contribution of

$$
d\left(m_1 c\frac{dx_i}{ds_1}\right) \text{ respectively } d\left(m_2 c\frac{dy_i}{ds_2}\right)
$$

and every contact line R'^i between dx^i and dy'^i gives a (potential) contribution

$$
\frac{e_1e_2}{8\pi c}d\left\{\frac{x_i}{(R'\cdot dx)} + \frac{dy'_i}{(R'\cdot dy')} - R'_i\frac{(dx\cdot dy')}{(R'\cdot dx)(R'\cdot dy')}\right\}. \tag{1}
$$

In case the motion is periodic, then it will be the case that the chain is also periodic, or at least almost periodic, and one will extend the mentioned sum over a single period, which makes the summation finite. It will then give a total differential and a conserved quantity will then exist which will not be changed when the zigzag-chain is displaced along the motions. When this constant quantity is divided by the number of contact lines that are counted in the finite period, then one will find the energy for $i = 0$ and the negatively taken quantity of motion for $i = 1, 2, 3$.

This is very much in accordance to the above mentioned remark that in this case the energy and quantity of motion cannot be defined for the system of particles, but only for the system of both complete motions.

In addition we have the circumstance that when the motions do not show any periodicity, that then the mentioned summation cannot be terminated to form a total differential. At the edges of the zigzag-chain some parts are then still missing. If one neglects these, and one again divides by the number of connected pairs in the chain, then it is true that the longer the chain, i.e. the longer the time one follows the motion, one can define with ever increasing precision the energy and momentum, but it remains in principle possible that the energy is not definable at a particular moment.

This is the drawback of the fact that the field is eliminated from the here formulated point-mechanical variation-law. But certainly, such a behaviour of energy and momentum does not contradict the quantum mechanical view.

Natuurkundig Laboratorium of Teyler's Stichting[4].

[4]Physical Laboratory of Teyler's Foundation.