The circularity in von Neumann's proof.

(Translation by Michiel Seevinck of "Der Zirkel in NEUMANNs Beweis", section 7 from the essay by Grete Hermann, *Die Naturphilosophischen Grundlagen de Quantenmechanik.* Abhandlungen der fries'schen Schule, **6**, 1935.)

Nowadays there is no lack of efforts to prove that such discoveries which would give a new opportunity for exact predictability of all measurements are in principle impossible. Very thoroughly constructed is the proof in Von Neumann's mathematical development of the formalism[1]. Nevertheless a deep study shows in this case as well that this mathematically sound argumentation introduces, without any legitimate ground, in its formal premises an equivalent proposition as the one to be proved. It is contained in the following premise: Suppose we have an ensemble of physical systems, with $\mathfrak{R}$ and $\mathfrak{S}$ physical quantities that can be measured on this ensemble; the expectation value of $\mathfrak{R}$ (Expt($\mathfrak{R}$)) is the average value of all measurement outcomes that will be obtained when measuring $\mathfrak{R}$ on all systems of the ensemble, and is also the value that is expected to be obtained when measuring $\mathfrak{R}$ on an arbitrary element of this ensemble. Von Neumann requires that for this expectation value-function Expt($\mathfrak{R}$), defined using an ensemble of physical systems and producing a number for every physical quantity, $\mathrm{Expt}(\mathfrak{R}+\mathfrak{S}) = \mathrm{Expt}(\mathfrak{R})$ $+ \mathrm{Expt}(\mathfrak{S})$. In words: *The expectation value of a sum of physical quantities is equal to the sum of the expectation values of both quantities.* With this assumption the proof of von Neumann either succeeds or fails.

For classical physics this requirement is trivial and also for those quantum mechanical observables that do not restrict each other in their measurability and for which no indeterminacy relations exist. For it is the case that for two of such quantities the value of their sum is nothing but the sum of the values that each of them separately obtains. From this we undoubtedly get the same relation between the expectation values of these quantities. Not trivial however is the relation for quantum mechanical quantities for which indeterminacy relations hold. In fact the sum of two such quantities is not even defined: Because a sharp measurement of one of them excludes sharp measurement of the other one and thus because both quantities can not have sharp values at the same time, the commonly used definition of the sum of two quantities breaks down.

Thus for the above determined notion of the sum of two, not jointly measurable quantities, the above mentioned equation requires a proof. Von Neumann provides this

[1]Neumann: "Mathematische Grundlagen der Quantenmechanik." Berlin 1930.

in two steps: Because every ensemble of physical states can be decomposed into sub-ensembles whose elements have identical wave functions, the requirement in question has only to be proved for ensembles whose elements fulfill the requirement of having identical wave functions. For these ensembles Von Neumann referred to the fact that, in accordance with the formalism, the symbol $(R\phi, \phi)$, which represents a number and is interpreted as the expectation value of the quantity $\mathfrak{R}$ in the state ϕ, obeys the following rule $((R + S)\phi, \phi) = (R\phi, \phi) + (S\phi, \phi)$. ($R$ and S signify the mathematical operators corresponding to $\mathfrak{R}$ and $\mathfrak{S}$ and ϕ signifies the wave function of the observed system.) From this rule Von Neumann concludes that for ensembles of systems with identical wave functions, and also for all ensembles, the sum rule for expectation values holds, even for such quantities that cannot be measured simultaneously.

The interpretation of the expression $(R\phi, \phi)$ is crucial for the entire proof. The point of view that it represents the expectation value of the quantity R for the system in states ϕ amounts –as shown by the formalism– to the same thing as the probability interpretation of the wave function. Thus the considerations related to this interpretation can be used here without any further elaboration: Until the proof of the impossibility of new characteristics is found, which is still to be given here, the expression $(R\phi, \phi)$ can only be taken to express the expectation value of the $\mathfrak{R}$-measurement of ensembles of physical systems that are *required to be* in the state ϕ; it has to remain an open issue, in order for it to stay applicable, whether or not this expectation value is the same in all sub-ensembles that can be distinghuised by arbitrary new characteristics. If this indeed remains open, then one can no longer deduce from the sum-rule which holds for $(R\phi, \phi)$, that also in these sub-ensembles the expectation value of the sum of physical quantities equals the sum of the expectation values. But with this a necessary step in Von Neumann's proof collapses. If one –just like Von Neumann– does not give up this step, then one has tacidly assumed the unproven presupposition that the elements of an ensemble of physical systems characterised by ϕ cannot have any distinghuising characteristics on which the outcome of $\mathfrak{R}$ is dependent. The impossibility of such characteristics is just the thesis to pe proved. The prove thus runs into a circularity.

However from the standpoint of Von Neumann's calculus one can object against this that in his calculus all physical quantities, by axiom, are required to correspond to certain hermitian operators on a Hilbert space and that this correspondence will be inevitably broken by the discovery of new characteristics that remove the current limits of predictability. Indeed, the content of every discovery that can be represented in operator calculus can only be determined by the structure of a wavefunction, which gives for non-simultaneously measurable quantities the broadening as determined by the indeterminacy relations and which is only applicable using the probability interpretation.

However using this consideration it is in no way possible to turn the *physical* question whether or not the ongoing physical research can lead to more accurate predictions than currently possible, into the not at all equivalent *mathematical* question whether or not such a discovery can only be represented using the tools of the quantum mechanical operator calculus. A conclusive physical reason is needed when, not only the up untill now known physical data, but also all future results of research are to be connected in correspondence with the axioms of this formalism. How can such a reason be found? That the formalism is currently verified, so that it is legitimite to view it as the correct mathematical description of the known coherence of nature, does not imply that also the up until now undiscovered lawlike coherence must have the same mathematical structure.