

## Bell-Type Inequalities for Partial Separability in $N$ -Particle Systems and Quantum Mechanical Violations

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We derive  $N$ -particle Bell-type inequalities under the assumption of partial separability, i.e., that the  $N$ -particle system is composed of subsystems which may be correlated in any way (e.g., entangled) but which are uncorrelated with respect to each other. These inequalities provide, upon violation, experimentally accessible sufficient conditions for full  $N$ -particle entanglement, i.e., for  $N$ -particle entanglement that cannot be reduced to mixtures of states in which a smaller number of particles are entangled. The inequalities are shown to be maximally violated by the  $N$ -particle Greenberger-Horne-Zeilinger states.

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Given the recent experimental interest in quantum correlations in 3- and 4-particle systems described in Refs. [1–4], and the probable extension to even larger number of particles, it becomes relevant to determine whether such correlations are due to full  $N$ -particle quantum entanglement and not just classical combinations of quantum entanglement of a smaller number of particles. We here derive a set of Bell-type inequalities that addresses this question by generalizing an analysis of Svetlichny [5] from 3-particle to  $N$ -particle systems.

Besides this experimental interest, the Bell-type inequalities here presented also address the fundamental question of whether Nature somehow limits the number of particles that can be fully entangled, that is to say, whether or not some form of partial separability holds. Note that this partial separability is distinct from the well-studied notion of full separability in Refs. [6–8]. In the former the subsets of the  $N$  particles form (possibly entangled) extended systems which, however, are all uncorrelated with respect to each other, whereas in the latter each particle is uncorrelated with respect to all others.

Our results differ from other results on “ $N$ -particle Bell-type inequalities,” such as the recent ones found in Refs. [6–8], in the following way. None of these treat partial separability in full generality. They assume either full separability of all the particles or full separability of some subset, and do not address the issue discussed here.

In this Letter the study of partial separability will be shown to result in new types of hidden variable theories and to give experimentally accessible conditions that deal with both the experimental and the fundamental questions mentioned above.

These experimentally accessible conditions are provided by the Bell-type inequalities that we derive from the assumption of partial separability. Upon violation they

are sufficient conditions for full  $N$ -partite entanglement, i.e., conditions to distinguish between  $N$ -particle states in which all  $N$  particles are entangled to each other and states in which only  $P$  particles are entangled (with  $P < N$ ). We also show that these inequalities are maximally violated by the  $N$ -partite Greenberger-Horne-Zeilinger (GHZ) states. Lastly, we end with some concluding remarks that compare our conditions to other similar conditions.

In order to derive our main results, imagine a system decaying into  $N$  particles which then separate into  $N$  different directions. At some later time we perform dichotomous measurements on each of the  $N$  particles, represented by observables  $A^{(1)}, A^{(2)}, \dots, A^{(N)}$ , with possible results  $\pm 1$ . Let us now make the following hypothesis of *partial separability*: An ensemble of such decaying systems consists of subensembles in which each one of the subsets of the  $N$  particles forms (possibly entangled) extended systems which, however, are uncorrelated to each other. Let us, for the time being, focus our attention on one of these subensembles, formed by a system consisting of two subsystems of  $P < N$  and  $N - P < N$  particles which are uncorrelated to each other. Assume also, for the time being, that the first subsystem is formed by particles  $1, 2, \dots, P$  and the other by the remaining particles. We express our partial separability hypothesis by assuming a factorizable expression for the probability  $p(a_1 a_2 \dots a_N)$  for observing the results  $a_i$ , for the observables  $A^{(i)}$ :

$$p(a_1 a_2 \dots a_N) = \int q(a_1 \dots a_P | \lambda) \times r(a_{P+1} \dots a_N | \lambda) d\rho(\lambda), \quad (1)$$

where  $q$  and  $r$  are probabilities conditioned to the hidden variable  $\lambda$  with probability measure  $d\rho$ . Formulas similar to (1) with different choices of the composing particles and a different value of  $P$  describe the other subensembles.

We need not consider decomposition into more than two subsystems as then any two can be considered jointly as parts of one subsystem still uncorrelated with respect to the others. Although (1) expresses a hidden variable model for the local (i.e., uncorrelated) behavior of the two subsystems in relation to each other, we shall show that the same inequality derived below can be used to distinguish, in the quantum mechanical case, between fully entangled states and those only partially entangled.

Consider the expected value of the product of the observables in the original ensemble

$$\begin{aligned} E(A^{(1)}A^{(2)}\cdots A^{(N)}) &= \langle A^{(1)}A^{(2)}\cdots A^{(N)} \rangle \\ &= \sum_J (-1)^{n(J)} p(J), \end{aligned} \quad (2)$$

where  $J$  stands for an  $N$ -tuple  $j_1, \dots, j_N$  with  $j_k = \pm 1$ ,  $n(J)$  is the number of  $-1$  values in  $J$ , and  $p(J)$  is the probability of achieving the indicated values of the observables. Using the hypothesis of Eq. (1) as a constraint we now derive nontrivial inequalities satisfied by the numbers  $E(A^{(1)}A^{(2)}\cdots A^{(N)})$  when introducing two alternative dichotomous observables  $A_1^{(i)}, A_2^{(i)}$ ,  $i = 1, 2, \dots, N$  for each of the particles. To simplify the notation we write  $E(i_1 i_2 \cdots i_N)$  for  $E(A_{i_1}^{(1)} A_{i_2}^{(2)} \cdots A_{i_N}^{(N)})$ . For any value of  $P$  and any choice of these  $P$  particles to comprise one of the subsystems we obtain (proof in the Appendix) the following inequalities:

$$\sum_I \nu_{t(I)}^\pm E(i_1 i_2 \cdots i_N) \leq 2^{N-1}, \quad (3)$$

where  $I = (i_1, i_2, \dots, i_N)$ ,  $t(I)$  is the number of times

index 2 appears in  $I$ , and  $\nu_k^\pm$  is a sequence of signs given by

$$\nu_k^\pm = (-1)^{k(k\pm 1)/2}. \quad (4)$$

These sequences have period four with cycles  $(1, -1, -1, 1)$  and  $(1, 1, -1, -1)$ . We call these inequalities *alternating*. They are direct generalizations of the tripartite inequalities in Svetlichny [5]. The alternating inequalities are satisfied by a system with *any* form of partial separability, so their violation is a sufficient indication of full nonseparability.

Introduce now the operator

$$S_N^\pm = \sum_I \nu_{t(I)}^\pm A_{i_1}^{(1)} \cdots A_{i_N}^{(N)}. \quad (5)$$

Using Eq. (3) the  $N$ -particle alternating inequalities can be expressed as

$$|\langle S_N^\pm \rangle| \leq 2^{N-1}. \quad (6)$$

For even  $N$  the two inequalities are interchanged by a global change of labels 1 and 2 and are thus equivalent. However, for odd  $N$  this is not the case and thus they must be considered *a priori* independent. To see this consider the effect of such a change upon the cycle  $(1, -1, -1, 1)$ . If  $N$  is even, we obtain  $(-1)^{N/2}(1, 1, -1, -1)$  which gives the second alternating inequality. For  $N$  odd, we obtain  $\pm(1, -1, -1, 1)$ , which results in the same inequality. Similar results hold for the other cycle.

The two alternating solutions for  $N = 2$  are the usual Bell inequalities; the ones for  $N = 3$  give rise to the two inequalities found in Svetlichny [5,9] and for  $N = 4$  we have

$$\begin{aligned} &|E(1111) - E(2111) - E(1211) - E(1121) - E(1112) - E(2211) - E(2121) - E(2112) - E(1221) - \\ &E(1212) - E(1122) + E(2221) + E(2212) + E(2122) + E(1222) + E(2222)| \leq 8, \end{aligned}$$

where the second inequality is found by interchanging the observable labels 1 and 2.

The  $N$ -particle alternating inequalities were derived for hidden variable states  $\lambda$ . However, they also hold for  $N$ -partite quantum states which are  $(N - 1)$  partite entangled (or nonentangled). In order to see this, suppose we choose the set of all hidden variables to be the set of all states on the Hilbert space  $\mathcal{H}$  of the system and  $\rho(\lambda) = \delta(\lambda - \lambda_0)$ , where  $\lambda_0$  is a quantum state in which one particle (say, the  $N$ th) is independent from the others, i.e.,

$$\rho = \rho^{\{1, \dots, N-1\}} \otimes \rho^{\{N\}}. \quad (7)$$

We then recover the factorizable condition of Eq. (1):

$$p(a_1 a_2 \cdots a_N | \lambda_0) = p_{\rho^{\{1, \dots, N-1\}}}(a_1 a_2 \cdots a_{N-1}) p_{\rho^{\{N\}}}(a_N), \quad (8)$$

where  $p_{\rho^{\{1, \dots, N-1\}}}(a_1 a_2 \cdots a_{N-1})$  and  $p_{\rho^{\{N\}}}(a_N)$  are the corresponding (joint) quantum mechanical probabilities to obtain  $a_1, a_2, \dots, a_N$  for measurements of ob-

servables  $A^{(1)}, A^{(2)}, \dots, A^{(N)}$ . The expectation value  $E(A^{(1)}A^{(2)}\cdots A^{(N)})$  then becomes the quantum mechanical expression,  $E_{\lambda_0}(A^{(1)}A^{(2)}\cdots A^{(N)}) = \text{Tr}[\rho^{\{1, \dots, N-1\}} A^{(1)} \otimes A^{(2)} \otimes \cdots \otimes A^{(N-1)}] \text{Tr}[\rho^{\{N\}} A^{(N)}]$ . Thus the same bound as in the alternating inequalities of Eq. (6) also holds for the quantum mechanical expectation values for a state of the form Eq. (7).

Since the alternating inequalities of Eq. (6) are invariant under a permutation of the  $N$  particles, this bound also holds for a state in which a particle other than the  $N$ th factorizes, and, since the inequalities are convex as a function of  $\rho$ , it holds also for mixtures of such states. Hence, for every  $(N - 1)$ -particle entangled state  $\rho$ , we have

$$|\langle S_N^\pm \rangle_\rho| = |\text{Tr}(\rho S_N^\pm)| \leq 2^{N-1}. \quad (9)$$

Thus, a sufficient condition for full  $N$ -particle entanglement is a violation of Eq. (9).

Now from this it follows that, by using the inequalities of Eq. (9), one can experimentally address the fundamental question of whether there is a limit to the number of particles that can be fully entangled, i.e., whether or not all forms of partial separability can be excluded when increasing the number of particles  $N$ .

The maximal quantum mechanical violation on the left-hand side of the  $N$ -particle alternating inequalities of Eq. (6) is obtained for fully entangled  $N$ -particle quantum states and is equal to  $2^{N-1}\sqrt{2}$ . To see this, note that the following recursive relation holds:

$$S_N^\pm = S_{N-1}^\pm A_1^{(N)} \mp S_{N-1}^\mp A_2^{(N)}. \quad (10)$$

Consider the term  $S_{N-1}^\pm A_1^{(N)}$  which is a self-adjoint operator. The maximum  $K$  of the modulus of its quantum expectation  $|\langle S_{N-1}^\pm A_1^{(N)} \rangle|$  is equal to the maximum of  $|\langle S_{N-1}^\pm \rangle|$  since the eigenvalues of  $A_1^{(N)}$  are  $\pm 1$ , similarly for the other term. Thus one can take the  $N$ -particle bound as twice the  $(N-1)$ -particle bound. Since the quantum mechanical bound on the Bell inequality is  $2\sqrt{2}$ , the result follows.

This upper bound is in fact achieved for the GHZ states for appropriate values of the polarizer angles of the relevant spin observables. Consider the general GHZ state:

$$\begin{aligned} \Psi_N &= \frac{1}{\sqrt{2}} (|\uparrow\rangle^{\otimes N} \pm |\downarrow\rangle^{\otimes N}) \\ &= \frac{1}{\sqrt{2}} (|\uparrow\uparrow\cdots\uparrow\rangle \pm |\downarrow\downarrow\cdots\downarrow\rangle). \end{aligned}$$

Let  $A_i^{(k)} = \cos\alpha_i^{(k)}\sigma_x + \sin\alpha_i^{(k)}\sigma_y$  denote spin observables with angle  $\alpha_i^{(k)}$  in the  $x$ - $y$  plane. A simple calculation shows

$$E(i_1 \cdots i_N) = \pm \cos(\alpha_{i_1}^{(1)} + \cdots + \alpha_{i_N}^{(N)}), \quad (11)$$

where the sign is the sign chosen in the GHZ state.

We now note that, for  $k = 0, 1, 2, \dots$ , one has  $\cos(\pm\frac{\pi}{4} + k\frac{\pi}{2}) = \nu_k^\pm \frac{\sqrt{2}}{2}$ , where  $\nu_k^\pm$  is given by (4). This means that, by a proper choice of angles, we can match, up to an overall sign, the sign of the cosine in Eq. (11) with the sign in front of  $E(i_1 \cdots i_N)$  as it appears in the inequality, forcing the left-hand side of the inequality to be equal to  $2^{N-1}\sqrt{2}$ . This can be easily done if, each time an index  $i_j$  changes from 1 to 2, the argument of the cosine is increased by  $\frac{\pi}{2}$ . Choose therefore

$$\begin{aligned} (\alpha_1^{(1)}, \alpha_1^{(2)}, \dots, \alpha_1^{(N)}) &= \left( \pm \frac{\pi}{4}, 0, \dots, 0 \right), \\ (\alpha_2^{(1)}, \alpha_2^{(2)}, \dots, \alpha_2^{(N)}) &= \left( \pm \frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2} \right), \end{aligned}$$

where the sign indicates which of the two  $\nu^\pm$  inequalities is used.

Recently Seevinck and Uffink [10] argued that the experimental data from some recent experiments (Refs. [1,2,4]) designed to produce full 3-particle entangled states do not completely rule out the hypothesis

of a partially entangled state. Further, an analysis of these experiments shows that the 3-particle alternating inequalities presented above would not be violated by the choice of the experimental observables and thus, based on the present inequalities, full entanglement in these experiments is still not established. However, we hope that future experiments (including  $N = 4$  and higher) will yield experimental tests of the alternating inequalities and will thereby provide conclusive tests for the existence of full  $N$ -partite entanglement. See also Cereceda [11] for another analysis of this point.

Similar conditions as the ones presented here to test for full entanglement were obtained in Gisin and Bechmann-Pasquinucci [6]. However, these differ in at least two aspects. First, for  $N$  even they are equivalent, whereas for  $N$  odd this is not the case (see Uffink [12]). Second and more important, the inequalities of Ref. [6] are derived only quantum mechanically, i.e., they only hold for quantum systems, and are thus unable to address the general requirement of partial separability which has been treated here.

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*Appendix: Proof of inequality (3).*—We seek inequalities of the form

$$\sum_I \sigma_I E(i_1 i_2 \cdots i_N) \leq M, \quad (12)$$

where  $\sigma_I$  is a sign and  $M$  is nontrivial. Following almost verbatim the analysis in [5], one must look for  $\sigma_I$  which solve the minimax problem

$$m = \min_\sigma m_\sigma = \min_\sigma \max_{\xi, \eta} \sum_I \sigma_I \xi_{i_1 \cdots i_P} \eta_{i_{P+1} \cdots i_N}, \quad (13)$$

where  $\xi_{i_1 \cdots i_P} = \pm 1$  and  $\eta_{i_{P+1} \cdots i_N} = \pm 1$  are also signs. Without loss of generality we can take  $P \geq N - P$ .

One can derive some useful upper bound on  $m$ . Toward this end, set  $\eta_{i_{P+1} \cdots i_{N-2}} = \zeta_{i_{P+1} \cdots i_{N-1}} \eta_{i_{P+1} \cdots i_{N-1}}$  for some sign  $\zeta_{i_{P+1} \cdots i_{N-1}}$ , using the fact that  $i_N = 1, 2$ . Taking into account that  $\sigma_I^2 = 1$ , and denoting by  $\hat{I}$  the  $(N-1)$ -tuple  $(i_1, \dots, i_{N-1})$  we have

$$\begin{aligned} m_\sigma &= \max_{\hat{I}} \sum_I \sigma_I \eta_{i_{P+1} \cdots i_{N-1}} \xi_{i_1 \cdots i_P} \\ &\quad \times (1 + \sigma_I \sigma_{\hat{I}2} \zeta_{i_{P+1} \cdots i_{N-1}}), \end{aligned}$$

the maximum being over  $\xi_{i_1 \cdots i_P}$ ,  $\eta_{i_{P+1} \cdots i_{N-1}}$ , and  $\zeta_{i_{P+1} \cdots i_{N-1}}$ .

Now certainly one has

$$m_\sigma \leq \hat{m}_\sigma = \max_{\hat{I}} |1 + \sigma_{\hat{I}1} \sigma_{\hat{I}2} \zeta_{i_{P+1} \cdots i_{N-1}}|, \quad (14)$$

the maximum taken over  $\zeta_{i_{P+1} \cdots i_{N-1}}$ .

If we define  $\hat{m} = \min_{\sigma} \hat{m}_{\sigma}$ , one easily sees that  $\hat{m} = 2^{N-1}$ . This can only be achieved under the following condition:

For each fixed  $(i_{P+1}, \dots, i_{N-1})$  exactly  $2^{P-1}$  of the quantities  $\sigma_{i_1} \sigma_{i_2}$  are  $+1$  and  $2^{P-1}$  are  $-1$ . (15)

Although it may be that  $m < \hat{m}$  we have proven that  $m = \hat{m} = 8$  in all cases for  $N = 4$ , and  $m = \hat{m}$  for  $P = N - 1$  for any  $N$ .

We shall call any choice of the  $\sigma_I$  satisfying this condition a *minimal solution*.

What immediately follows from the above is that any solution of (15) for a given value of  $P$  is a solution for all greater values of  $P \leq N - 1$ . A violation of an inequality so obtained for the smallest possible value of  $P \geq N/2$  precludes then any factorizable model of the  $N$ -particle correlations.

Assume provisionally that the only decays are those in which an  $N = P + (N - P)$  factorization occurs. The whole ensemble of decays consists of subensembles corresponding to different choices of the  $P$  particles. We do not know in any particular instance of decay to which of the subensembles the event belongs. To take account of this, our inequality must be one that would arise under any choice of the  $P$  particles. Call a minimal solution  $\sigma_I$  *admissible* if  $\sigma_{\pi(I)}$  is also a minimal solution for any permutation  $\pi$ . An inequality that follows from an admissible solution will therefore be one that must be satisfied by any subensemble of the  $N = P + (N - P)$  factorizable events.

The set of admissible solutions breaks up into orbits by the action of the permutation group. The overall sign of  $\sigma_I$  is not significant and two solutions that differ by a sign are considered equivalent. The set of these equivalence classes also breaks up into orbits by the action of the permutation group. It is remarkable that there are orbits consisting of one equivalence class only. For such, one must have  $\sigma_{\pi(I)} = \pm \sigma_I$ . The sign in front of the right-hand side must be a one-dimensional representation of the permutation group, so one must have either  $\sigma_{\pi(I)} = \sigma_I$  or  $\sigma_{\pi(I)} = (-1)^{s(\pi)} \sigma_I$ , where  $s(\pi)$  is the parity of  $\pi$ . The second case is impossible since one then would have  $\sigma_{11\dots 1} = -\sigma_{11\dots 1}$  as a result of a flip permutation. Since an overall sign is not significant, one can now fix  $\sigma_{11\dots 1} = 1$ . Since the only permutation invariant of  $I$  is  $t(I)$ , the number of times index 2 appears in  $I$ , we must have  $\sigma_I = \nu_{t(I)}$  for some  $(N + 1)$ -tuple ( $\nu_0 = 1$  by convention)  $\nu = (1, \nu_1, \nu_2, \dots, \nu_N)$ . We must now solve for the possible values of  $\nu$ .

Let  $a = t(i_{P+1} \dots i_{N-1})$  and  $b = t(i_1 \dots i_P)$ ; then condition (15) for our choice of  $\sigma_I$  is equivalent to  $\nu$ , satisfy-

ing

$$\sum_{b=0}^P \binom{P}{b} \nu_{a+b} \nu_{a+b+1} = 0, \quad a = 0, 1, \dots, N - P - 1. \quad (16)$$

Let  $\mu_k = \nu_k \nu_{k+1}$ . Equation (16) then becomes

$$\sum_{b=0}^P \binom{P}{b} \mu_{a+b} = 0, \quad a = 0, 1, \dots, N - P - 1. \quad (17)$$

Now it is obvious that there are at least two solutions of (17) valid for all  $P$ , to wit,  $\mu_k = \pm(-1)^k$  since then (17) is just the expansion of  $(1 - 1)^P$  or  $(-1 + 1)^P$ . Call these solutions the *alternating solutions*. Finally, we obtain from  $\mu_k = \nu_k \nu_{k+1}$  the two solutions (4) once we have chosen the overall sign to set  $\nu_0 = 1$ .

*Note added.*—After this paper was submitted for publication, an article by Collins *et al.* [13] appeared that treats some of the same questions and gives independent proof of our alternating inequalities.

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