Bell Inequality (Non-)Violation versus Local (Non-) Commutativity [1]

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I: Bell inequality & Local commutativity

Consider the well known Bell operator:

$$\mathcal{B} := A \otimes (B + B') + A' \otimes (B - B'). \quad (1)$$

For the set of separable states $\mathcal{D}_{\rm sep}$ we have $|\langle \mathcal{B}\rangle_\rho| \leq 2$, whereas for the set of all (possibly entangled) quantum states $\mathcal{D}$ we get the Tsirelson inequality:

$$|\langle \mathcal{B}\rangle_\rho| \leq \sqrt{4 + |\langle [A, A'] \otimes [B', B]\rangle_\rho|} \leq 2\sqrt{2}. \quad (2)$$

Consider qubits on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and general spin observables, e.g. $A = \boldsymbol{a}\cdot\boldsymbol{\sigma} = \sum_i a_i\sigma_i$. Denote by θ_A the angle between A and A' (i.e., $\cos\theta_A = \boldsymbol{a}\cdot\boldsymbol{a}'$) and analogously for θ_B. Then we have:

- Local commutativity ($[A, A'] = [B, B'] = 0$) implies $\theta_A = \theta_B = 0$ (mod. π), i.e. parallel observables.
- Local anticommutativity ($\{A, A'\} = \{B, B'\} = 0$) implies $\theta_A, \theta_B = \pm\pi/2$, i.e. orthogonal observables.

Maximal violation $\Longrightarrow$ local anti-commutativity

To get maximal violation of (2) we need:

$$|\langle [A, A'] \otimes [B', B]\rangle_\rho| = 4|\langle (\boldsymbol{a}\times\boldsymbol{a}')\cdot\boldsymbol{\sigma} \otimes (\boldsymbol{b}\times\boldsymbol{b}')\cdot\boldsymbol{\sigma}\rangle_\rho| = 4.$$

This can equal 4 only if $||\,\boldsymbol{a}\times\boldsymbol{a}'|| = ||\,\boldsymbol{b}\times\boldsymbol{b}'|| = 1$, which implies that $\boldsymbol{a}\cdot\boldsymbol{a}' = 0$ and $\boldsymbol{b}\cdot\boldsymbol{b}' = 0$.

Thus maximal violation is only possible if the local observables are orthogonal, i.e., $\theta_A = \theta_B = \pi/2$, and to get any violation at all it is necessary that the local observables are at some angle to each other, i.e., $\theta_A \neq 0,\ \theta_B \neq 0$.

Inspired by this, we seek a trade-off relation that expresses exactly how the amount of violation depends on the local angles $\theta_A,\ \theta_B$ between the spin observables. We thus seek the form of

$$C(\theta_A, \theta_B) := \max_{\rho\in\mathcal{D}} |\langle \mathcal{B}\rangle_\rho| \quad (3)$$

Local anti-commutativity and separable states

For separable states $\rho \in \mathcal{D}_{\rm sep}$ and local orthogonal observables the following separability inequality holds [2]:

$$\langle \mathcal{B}\rangle_\rho^2 + \langle \mathcal{B}'\rangle_\rho^2 \leq 2\langle \mathbb{1}\otimes\mathbb{1} - A''\otimes B''\rangle_\rho^2 - 2\langle A''\otimes\mathbb{1} - \mathbb{1}\otimes B''\rangle_\rho^2, \quad (4)$$

with the $A'' = i[A, A']/2$ and $B'' = i[B, B']/2$ and where $\mathcal{B}'$ is the same as $\mathcal{B}$ but with $A \leftrightarrow A'$, $B \leftrightarrow B'$. Note the strength of (4). If it holds for all sets of local orthogonal observables it is necessary and sufficient for separability [2].

From (4) we get the following separability inequality for all states in $\mathcal{D}_{\rm sep}$:

$$|\langle \mathcal{B}\rangle_\rho| \leq \sqrt{2(1 - \frac{1}{4}|\langle [A, A']\rangle_{\rho_1}|^2)(1 - \frac{1}{4}|\langle [B, B']\rangle_{\rho_2}|^2)}, \quad (5)$$

where ρ_1 and ρ_2 are the single qubit states.

The inequality (5) is the separability analogue for anti-commuting observables of the Tsirelson inequality (2). Note that even in the weakest case ($\langle [A, A']\rangle_{\rho_1} = \langle [B, B]\rangle_{\rho_2} = 0$) it implies $|\langle \mathcal{B}\rangle_\rho| \leq \sqrt{2}$, which strengthens the original Bell-CHSH inequality.

Thus we see a reversed effect: in contrast to entangled states, the requirement of anticommutivity (i.e., local orthogonality of the observables) thus decreases the maximum expectation value of $\mathcal{B}$ for separable states.

Inpired by this, we look for a trade-off relation that expresses exactly how the maximum bound for $\langle \mathcal{B}\rangle_\rho$ depends on the local angles of the spin observables in the case of separable states. We thus seek the form of

$$D(\theta_A, \theta_B) := \max_{\rho\in\mathcal{D}_{\rm sep}} |\langle \mathcal{B}\rangle_\rho|. \quad (6)$$

II: Tradeoff relations

General qubit states

$$C(\theta_A, \theta_B) = \sqrt{4 + 4|\sin\theta_A \sin\theta_B|}. \quad (7)$$

This is plotted in Fig 1.

If both angles are chosen the same, i.e., $\theta_A = \theta_B := \theta$, (7) simplifies to

$$C(\theta, \theta) = \sqrt{4 + 4\sin^2\theta}, \quad (8)$$

which is plotted in Fig 3.

Separable states

$$D(\theta_A, \theta_B) = \Big|\mathrm{W}_+(1 + \mathrm{X}_\pm^2)^{-1/2} + \cos(\arctan(\mathrm{X}_\pm) - \theta_A)\,\mathrm{W}_-\Big|, \quad (9)$$

with $\mathrm{W}_\pm, \mathrm{X}_\pm, \mathrm{Y}$ and Z complicated functions of $\theta_A,\ \theta_B$, see [1]. The function (9) is plotted in Fig.2.

As a special case, suppose we choose $\theta_A = \theta_B := \theta$. Then, (9) reduces to the much simpler expression

$$D(\theta, \theta) = \cos\theta + \sqrt{1 + \sin^2\theta}. \quad (10)$$

This result strenghtens the bound obtained previously by Roy [3] for this special case. Both bounds are shown in Fig 3.

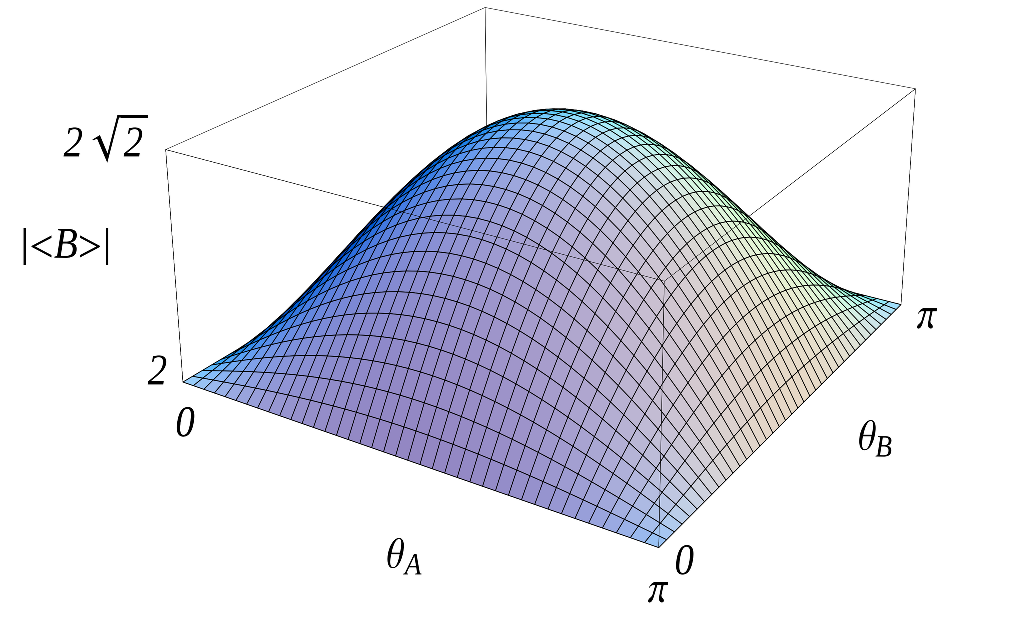

Fig. 1: Plot of $C(\theta_A, \theta_B) = \max_{\rho\in\mathcal{D}} |\langle \mathcal{B}\rangle_\rho|$ as given in (7) for $0 \leq \theta_A, \theta_B \leq \pi$.

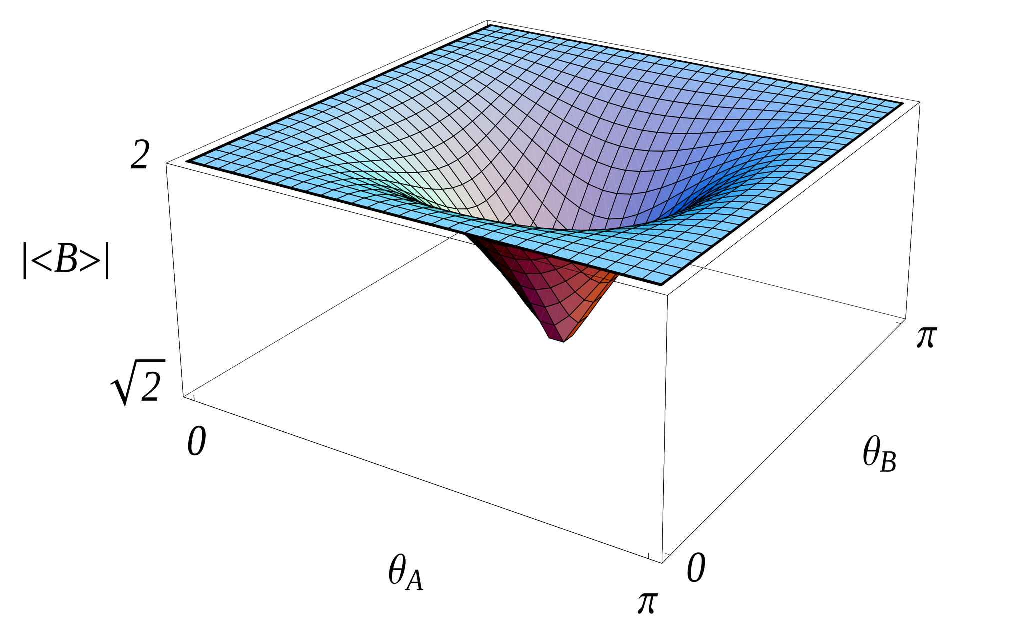

Fig. 2: Plot of $D(\theta_A, \theta_B) := \max_{\rho\in\mathcal{D}_{\rm sep}} |\langle \mathcal{B}\rangle_\rho|$ as given in (9) for $0 \leq \theta_A, \theta_B \leq \pi$.

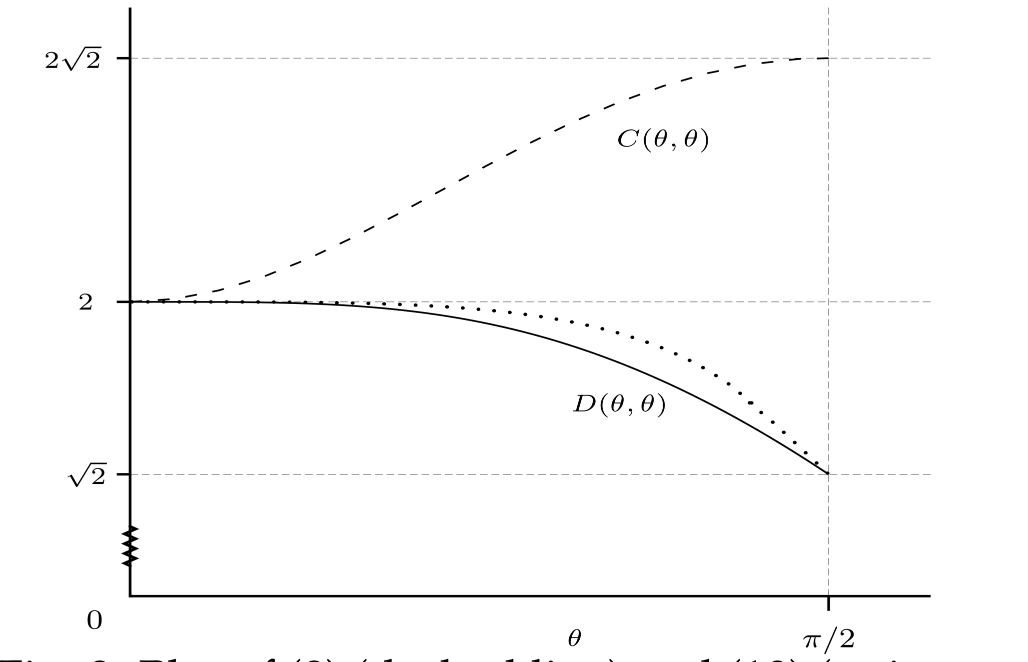

Fig. 3: Plot of (8) (dashed line) and (10) (uninterupted line), and Roy's bound [3] (dotted line).

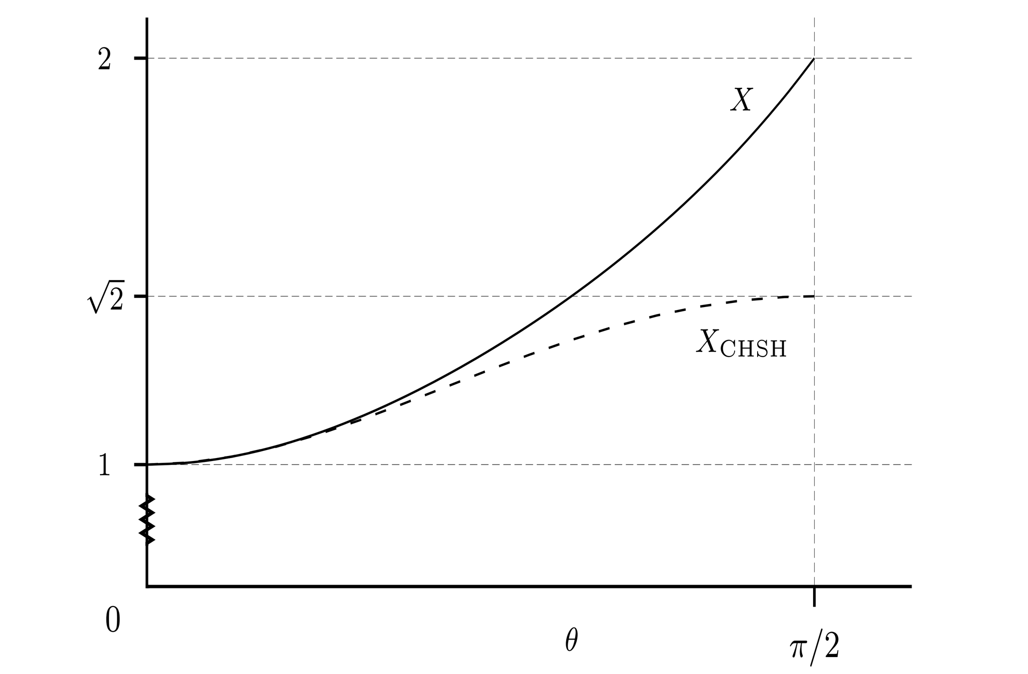

Fig. 4: Violation factor X (uninterupted line) and $X_{\rm CHSH}$ (dashed line) for $\theta_A = \theta_B := \theta$.

III: Discussion

We have obtained tight quantitative expressions for two trade-off relations [1].

(1): Between the degrees of local commutativity, as measured by the local angles θ_A and θ_B, and the maximal degree of Bell-CHSH inequality violation.

(2): Secondly, a converse trade-off relation holds for separable states: if both local angles increase towards $\pi/2$, the value obtainable for the expectation of the Bell operator decreases. (The non-violation of the Bell-CHSH ineqquality increases)

The extreme cases are obtained for anti-commuting (=orthogonal) local observables where the bounds of $2\sqrt{2}$ and $\sqrt{2}$ hold.

Foundational relevance

These two trade-off relations show that local non-commutativity has two diametrically opposed features:

On the one hand, the choice of locally non-commuting observables is necessary to allow for any violation of the Bell-CHSH inequality in entangled states (a "*more than classical*" result).

On the other hand, this very same choice of non-commuting observables implies a "*less than classical*" result for separable states: For such states the correlations obey a more stringent bound ($\langle \mathcal{B}\rangle_\rho \leq \sqrt{2}$) than allowed for in local hidden variable theories ($\langle \mathcal{B}_{\rm CHSH}\rangle \leq 2$).

Experimental relevance

The separability inequalities of Eq. (9) and (10) can be regarded as entanglement witnesses. They compare favourably to the Bell-CHSH inequality as a witness of entanglement. They furthermore allow for some uncertainty about the precise observables one is implementing.

Let us define the 'violation factor' X as the ratio $C(\theta_A, \theta_B)/D(\theta_A, \theta_B)$, i.e., the maximum correlation obtained by entangled states divided by the maximum corelation attainable for separable states. In Fig. 4 this is plotted for the of equal angles. This is compared to the ratio by which these maximal correlations violate the Bell-CHSH inequality, i.e. $X_{\rm CHSH} := C(\theta, \theta)/2$.

The results of Fig. 4 imply that the comparison of the maximum correlation in entangled states to the maximum correlations in separable states yields a stronger witness for entanglement than its comparison to the Bell-CHSH inequality. Thus, the separability inequalities (9) and (10) allow for greater noise robustness (cf. [2]).

References: [1] M. Seevinck and J. Uffink, quant-ph/0703134 (2007). [2] J. Uffink and M. Seevinck, quant-ph/0604145 (2006). [3] S.M. Roy, Phys. Rev. Lett. **94**, 010402 (2005).