Multi-partite entanglement

versus

Multi-partite non-locality

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∞ Joint work with George Svetlichny and Jos Uffink ∞

∞ Motivation ∞

- To clarify the **relationship** between the local hidden variable structure and quantum mechanics in a **multi-partite** setting.

  "Whereas in the late Eighties there was hardly any difference between entangled states and states violating a Bell inequality, we have a much more subtle discrimination nowadays." R. Werner (2001)

- **Fundamental**:
  1) Is a local realistic description of quantum phenomena possible?
  2) Does Nature somehow **limit** the number of particles that can be fully entangled, that is to say, does some form of partial separability holds?

- **Experimental**: Do we have **full** N-particle quantum entanglement and not just classical combinations of quantum entanglement of a smaller number of particles?

∞ Outline ∞

1. **Entanglement vs. non-locality in bi-partite systems:**

   - Entanglement (Quantum mechanics)

   - Factorisability and Bell inequalities (LHV theories)

2. **Extension to multi-partite setting:**

   - Full/partial entanglement (Quantum Mechanics)

   - Full/partial factorisability and Svetlichny inequalities (LHV Theories)

   For clarity only the **three-partite case** will be treated.

3. **Quadratic Bell-type inequalities**

4. **Conclusion and outlook**

∞ Formalism ∞

I Quantum Mechanics (QM)

- **Only spin 1/2 particles**, each on Hilbert space $\mathcal{H} = \mathbb{C}^2$.

- **Mixed (and pure) quantum mechanical states**:
  Density operators ρ.

- **Only orthodox measurements**:
  Every observable A corresponds to a self adjoint operator $\hat{A}$, and has expectation value $E^{QM}(A)_\rho =< A >_\rho = \mathrm{Tr}[\rho\hat{A}]$.

II Hidden Variables theory (HV)

- **Hidden variable space** Λ.

- **Pure states**: points $\lambda \in \Lambda$.

  Mixed states: distributions $\mu(\lambda)$.

- **Observables**: real valued functions from Λ to $\mathbb{R}$.

∞ Concepts ∞

(I) Quantum Mechanics (QM):

Consider a bi-partite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

- **Separable States**:

  Convex sums of direct-product states:

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \qquad \sum_i p_i = 1,\ p_i > 0.$$

  Example: Direct-product states:

$$\rho = \rho^A \otimes \rho^B.$$

- **Entangled states**:

  Non-separable states: $\rho \neq \sum_i p_i \rho_i^A \otimes \rho_i^B$.

  Example: The well-known singlet state:

$$|\,\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle).$$

(II) Local Hidden Variable Theory:

- **HV-model predictions for correlations** $\mathbb{P}(a,b|A,B)$**:**
  - There exist probability functions $p_{A,B}(a,b,\lambda)$ such that:

    $$\mathbb{P}(a,b|A,B) = \int_\Lambda d\lambda \; \mu(\lambda) \; p_{A,B}(a,b,\lambda)$$

    for some (mixed) hidden variable state $\mu(\lambda)$.

  - **Require locality** for both outcomes and parameter settings:

    $$p_{A,B}(a,b|\lambda) = p_A(a|\lambda) p_B(b|\lambda).$$

    This is factorisability: Outcomes a and b for a fixed λ are completely statistically independent.

  - Locality $\Longrightarrow$ Bell-inequality (Bell (1964), CHSH (1969))

    $$\left| E^{\rm lr}(A_1,B_1) + E^{\rm lr}(A_2,B_1) + E^{\rm lr}(A_1,B_2) - E^{\rm lr}(A_2,B_2) \right| \leq 2.$$

Note the **difference** between the quantum mechanical separability and factorisability in hidden variable theories.

∞ Results on Entanglement vs. Locality (bi-partite) ∞

- **All pure entangled states can be made to violate a Bell inequality** (Gisin and Peres (1996)):

  $\Longrightarrow$ No LHV exist for QM. (Bell (1964))

- **Werner states** (Werner 1989): 'Hidden non-locality', These are entangled states (mixed) but nevertheless allow for LHV model for orthodox measurements.

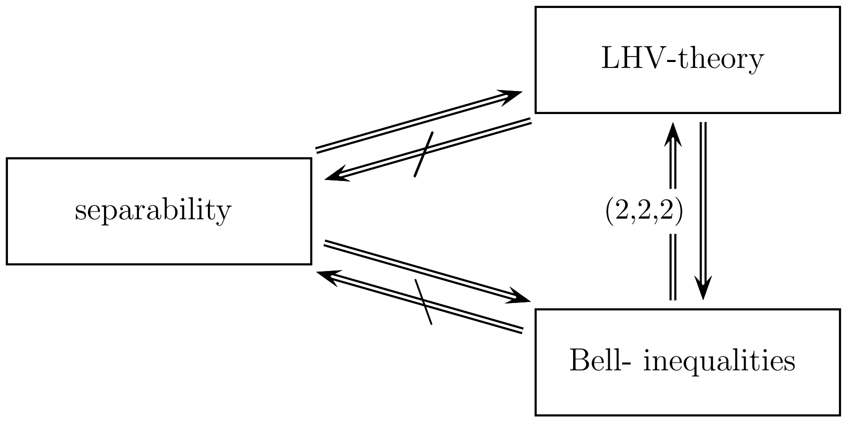
  - Entanglement does not rule out factorisability.

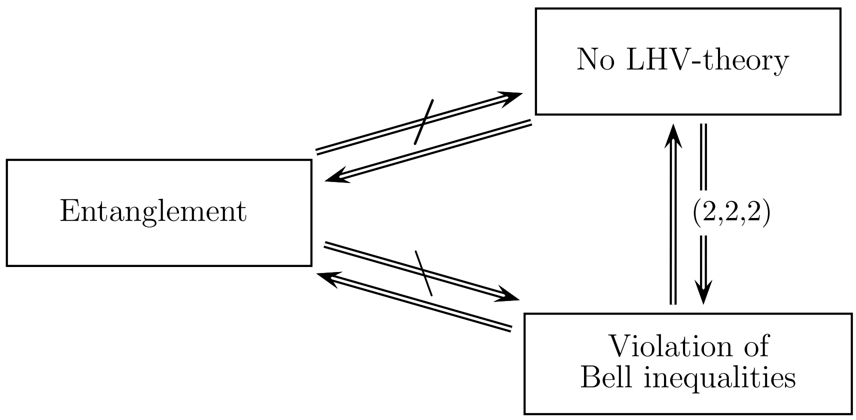
  - Separability $\Longrightarrow$ factorisability

    The converse does not hold.

  - Reveiling hidden non-locality using POVM measurements (Popescu (1996)).

- **Necessary and sufficient sets** of Bell inequalities have been found only for measurements of two dichotomous observables. (Zukowski, Werner and Wolff (2001))

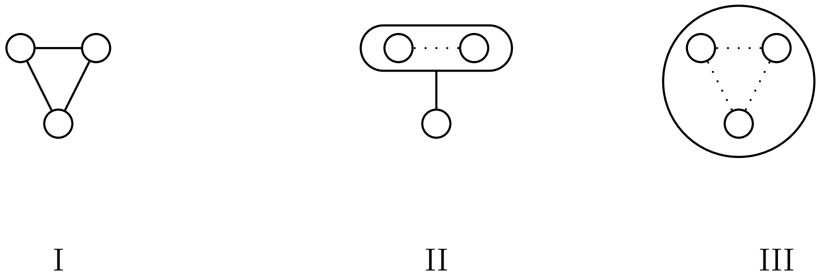

LHV-theory
separability
(2,2,2)
Bell- inequalities


No LHV-theory
Entanglement
(2,2,2)
Violation of
Bell inequalities

∞ Multi-partite ∞

∞ Concepts ∞

A: Quantum mechanics



I Unentangled states:

$$\rho = \sum_{ijk} p_{ijk} \rho_i \otimes \rho_j \otimes \rho_k, \qquad \text{for some } \rho_i,\ \rho_j,\ \rho_k \text{ on } \mathbb{C}^2.$$

II Partially entangled states:

$$\rho = p_a \rho^{(12)} \otimes \rho^{(3)} + p_b \rho^{(13)} \otimes \rho^{(2)} + p_c \rho^{(23)} \otimes \rho^{(1)}$$

III Fully entangled states:

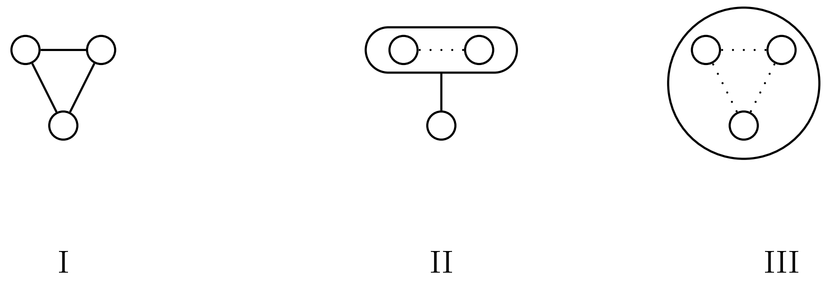
$$\rho \neq \sum_i p_i \rho_i,$$

with all the states ρ_i on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ separable into products of states of less than three parties.

- This **excludes** $\rho = \rho_A \otimes \rho_B \otimes \rho_C$ and $\rho = \rho_{AB} \otimes \rho_C$.

- **Example:** GHZ-state: $|\Psi_{GHZ}\rangle = (|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)/\sqrt{2}$

B: Hidden variables theory



I Full factorisability, for example:

$$p_{ABC}(a,b,c) = \int_\Lambda p_A(a|\lambda)p_B(b|\lambda)p_C(c|\lambda)\mu(\lambda)d\lambda.$$

II Partial factorisability, for example:

$$p_{ABC}(a,b,c) = \int_\Lambda p_{AB}(a,b|\lambda)p_C(c|\lambda)\mu(\lambda)d\lambda.$$

Thus the subsystems can be partitioned into groups of which the internal states can be correlated in any way (e.g. entangled), whereas the groups themselves behave factorisable (independent).

III No factorisability.

Partial Local Hidden Variables Theory (PLHV):
a hybrid model with a partial factorisability requirement such as in case II (or permutations).

∞ Partial factorisability implies Svetlichny Inequalities ∞

(This is analogous to: factorisability $\Longrightarrow$ Bell-inequalities.)

(A) Tri-partite (Svetlichny (1987))

- **Assumption: Partial factorisability**

$$\mathbb{P}(a,b,c|A,B,C) = \int_\Lambda p_{AB}(a,b|\lambda)\ p_C(c|\lambda)\ \mu(\lambda)d\lambda,$$

and all permutations of A, B, C.

- **Look at expectation values:**

$$E(ABC) = \sum_{abc} abc\ \mathbb{P}(a,b,c|A,B,C).$$

- **Wanted: Inequalities of the form**

$$\sum_{ijk} C_{ijk} E(A_i B_j C_k) \leq M, \qquad \text{with } M \text{ non} - \text{trivial } (M < 8).$$

- **Two Svetlichny inequalities** $S_3^\pm$ (All $\pm$ read the same)**:**

$$| \ E(ABC) \pm E(ABC') \pm E(AB'C) \pm E(A'BC) \\ -E(AB'C') - E(A'BC') - E(A'B'C) \pm E(A'B'C') \ | \ \leq \ 4.$$

- **These inequalities are necessary conditions for all three-partite PLHV-theories.**

Svetlichny inequalities as a test for full tri-partite entanglement.

- **Use:**
  - $E(\hat{A}, \hat{B}, \hat{C}) = \mathrm{Tr}[\hat{A} \otimes \hat{B} \otimes \hat{C} \otimes \rho]$.
  - State space $\Lambda = \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$
  - $\mu(\lambda) = \delta(\lambda - \lambda_0), \quad \lambda_0 = \rho^{(12)} \otimes \rho^{(3)}$

    Note: bi-separable restriction.

- **Then:**
  $E_{\lambda_0}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}[\lambda_0 \, \hat{A} \otimes \hat{B} \otimes \hat{C}] = \mathrm{Tr}[\rho^{(12)} \, \hat{A} \otimes \hat{B}]\mathrm{Tr}[\rho^{(3)} \, \hat{C}]$.

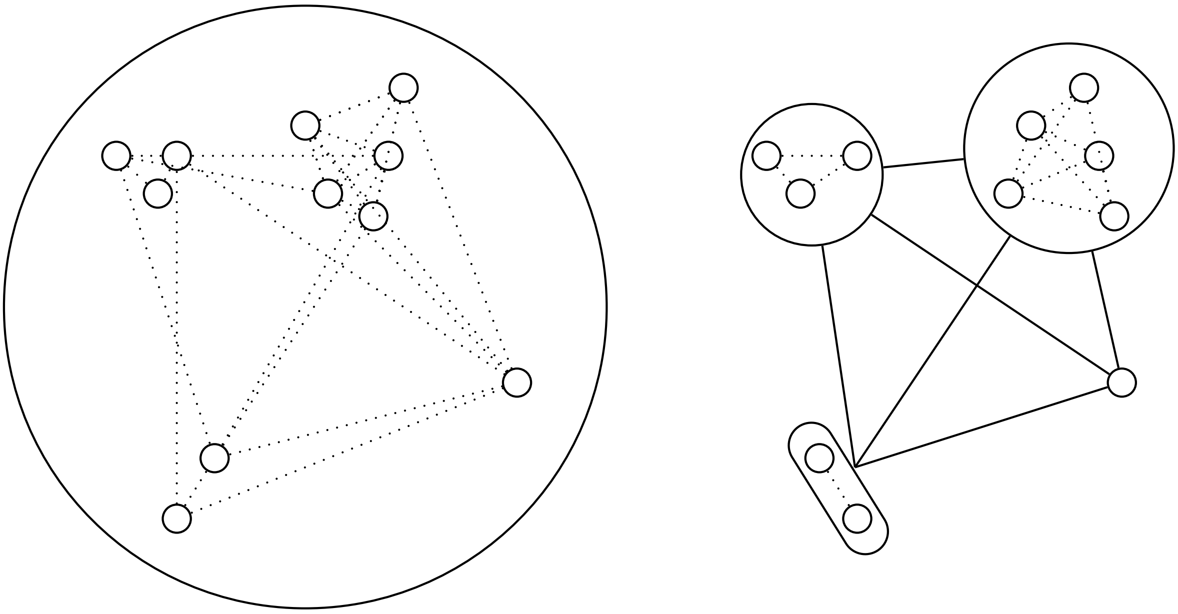
- **Implication:** Svetlichny inequalities hold for states of the form $\rho^{(12)} \otimes \rho^{(3)}$.

- **Also for:** All convex sums of bi-separable states, i.e. for all states which are **not** three-particle entangled.

- **Conclusion:** Violation of Svetlichny inequalities is a sufficient condition for three-particle entanglement

- **Maximum violation** is $4\sqrt{2}$. A ratio of $\sqrt{2}$.

(B) Multipartite generalizations



Key idea: Partial factorisability from 3-particle to N-particle systems.

Note: Other multi-partite inequalities (e.g. Mermin (1990)) only consider full factorisability.

Locality hypothesis of multipartite partial factorisability:

$$p(a_1 a_2 \cdots a_N) = \int q(a_1 \cdots a_P|\lambda) r(a_{P+1} \cdots a_N|\lambda)\, d\rho(\lambda) \tag{1}$$

Wanted: Inequalities of the form

$$\sum_I \sigma_I E(A^{(1)}_{i_1} \cdots A^{(N)}_{i_N}) \leq M, \qquad M \text{ non} - \text{trivial},$$

where $I = (i_1, i_2, \ldots, i_N)$ indicates which alternative observable was chosen for each particle, and $\sigma_I = \pm 1$ is a sign for each N-tuple I.

Use the operator:

$$S^{\pm}_N = \sum_I \nu^{\pm}_{t(I)} A^{(1)}_{i_1} \cdots A^{(N)}_{i_N}.$$

Result: Solving for $\nu^{\pm}_{t(I)}$ under the locality hypothesis gives the N-particle Svetlichny inequalities:

$$|\langle S^{\pm}_N \rangle| \leq 2^{N-1}.$$

$N = 2$ The CHSH-inequalities,

$N = 3$ Svetlichny inequalities for N=3

$N = 4$

$$|E(1111) + E(2111) + E(1211) + E(1121)+$$
$$E(1112) - E(2211) - E(2121) - E(2112) -$$
$$E(1221) - E(1212) - E(1122) - E(2221) -$$
$$E(2212) - E(2122) - E(1222) + E(2222)| \leq 8.$$

These inequalities are **a necessary condition** for **any** PLHV theory to hold. A violation excludes **all** multi-partite partial factorisations.

Note: No sufficient and necessary set of inequalities for a PLHV theory is known to exist.

These inequalities are a sufficient condition for N-partite entanglement:

Consider:

$$\rho = \rho^{\{1,\ldots,N-1\}} \otimes \rho^{\{N\}}. \tag{2}$$

Result: for every $(N-1)$-particle entangled state ρ:

$$|\langle S_N^{\pm} \rangle_\rho| = |\mathrm{Tr}(\rho S_N^{\pm})| \leq 2^{N-1}. \tag{3}$$

Conclusion: A **sufficient condition** for full N-particle entanglement is a violation of Eq.(3).

Maximal violation: $2^{N-1}\sqrt{2}$, Ratio: $\sqrt{2}$.

∞ Hidden nonlocality ∞

- Consider the following family of three-particle states:

$$\rho = a \, | \, \Psi_{\mathrm{GHZ}} \rangle \, \langle \, \Psi_{\mathrm{GHZ}} | + b \frac{\mathbb{I}}{8}$$

with $a + b = 1$ and $\frac{3}{7} \leq a \leq \frac{1}{\sqrt{2}}$.

- These states are **fully entangled** and can **not violate** the Svetlichny inequalities for any possible set of local measurements.

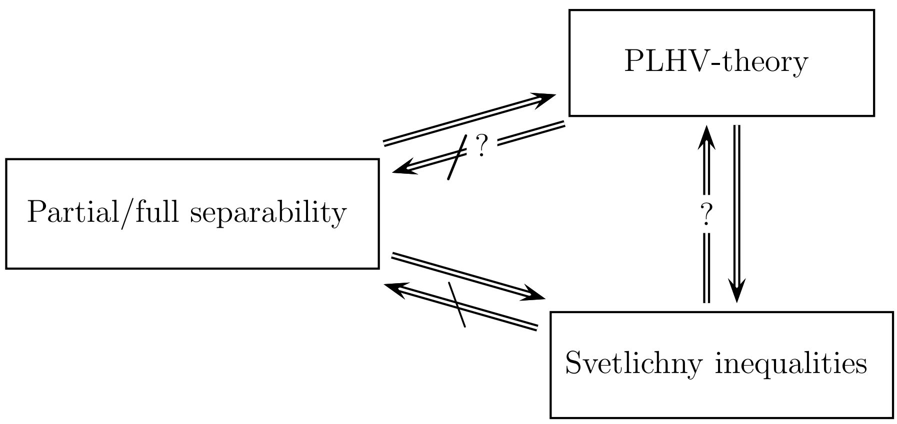
$\Longrightarrow$ **Partial factorisability and partial separability are distinct notions.**

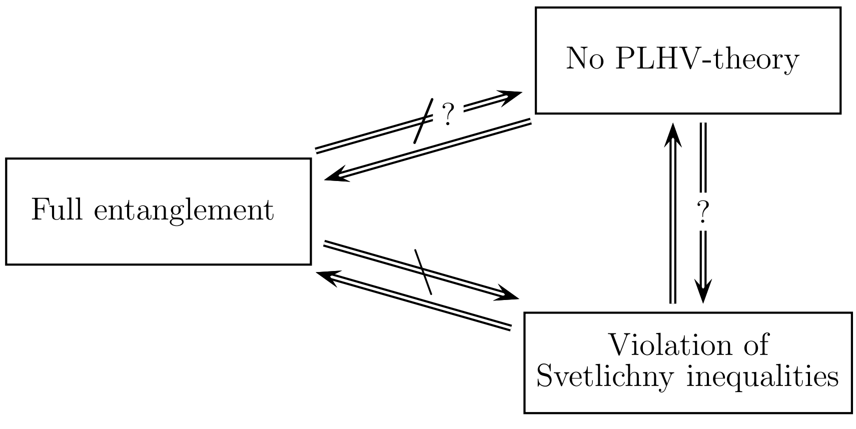
$\Longrightarrow$ **Full entanglement does not rule out partial factorisability.**

Analogous to the bi-partite case, i.e. the Werner states, although no explicit PLHV-model was constructed for these states.

∞ Conjecture ∞

The requirement of having a partially separable quantum description of three or more subsystems is at least as stringent a condition as the requirement of admitting any possible partial local hidden variable model, and further it is conjectured to be a strictly more stringent condition.





Quadratic inequalities

- Trivial: for all ρ:

$$
\begin{aligned}
| < AB + A'B + AB' - A'B' > | &\leq 4, \\
| < AB + AB' > | &\leq 2, \\
| < AB' - A'B' > | &\leq 2
\end{aligned}
$$

- Bell (1964): For all separable ρ:

$$| < AB + A'B + AB' - A'B' > | \leq 2$$

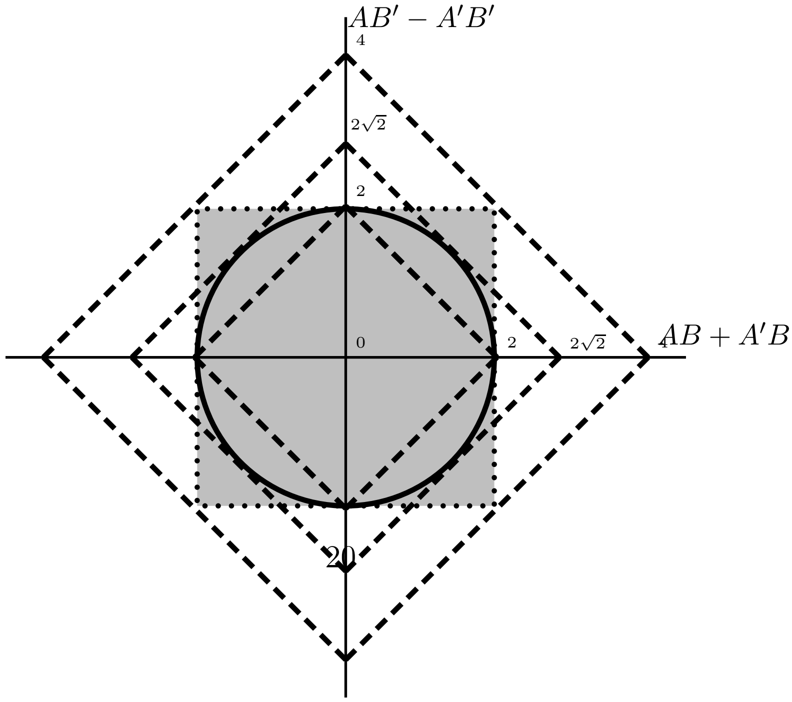
- Cirel'son (1980): For all ρ:

$$| < AB + A'B + AB' - A'B' > | \leq 2\sqrt{2}$$

- Uffink (2002): For all ρ:

$$< A'B + A'B >^2 + < AB - A'B' >^2 \leq 4$$

This is stronger than the Cirel'son bound.



Multipartite

Recursive method: Let A_j and A'_j denote dichotomous observables on the j-th particle.

$$F_N := \frac{1}{2}(A_N + A'_N)F_{N-1} + \frac{1}{2}(A_N - A'_N)F'_{N-1} \leq 2, \tag{4}$$

Belinskii-Klyshko inequality:

$$|E^{\mathrm{lr}}(F_N)| \leq 2, \tag{5}$$

Violation in quantum mechanics:

$$| < \widehat{F}_N > | \leq 2^{(N+1)/2}$$

Extension into a test of $N-1$-entanglement:

- Consider: $\rho = \rho_{\{N\}} \otimes \rho_{\{1,\ldots,N-1\}}$.
- Then for every $(N-1)$-particle entangled state:

$$| < \widehat{F}_N > | \leq 2^{N/2}. \tag{6}$$

- **Sufficient condition** for N-particle entanglement is a violation of (6).

Comparison of Belinskii-Klyshko, Svetlichny and Uffink inequality, Tri-partite case

Note:

$$S_3^{\pm} = \mp F_3 - F_3'.$$

Then for **all separable and bi-separable** ρ

$$| < S_3^{\pm} > | = | < F_3 \pm F_3' > | \leq 4.$$

Or equivalently:

$$\max | < F_3 > |, | < F_3' > | \leq 2\sqrt{2}.$$

The **quadratic inequality** reads:

$$< F_3 >^2 + < F_3' >^2 \leq 8$$

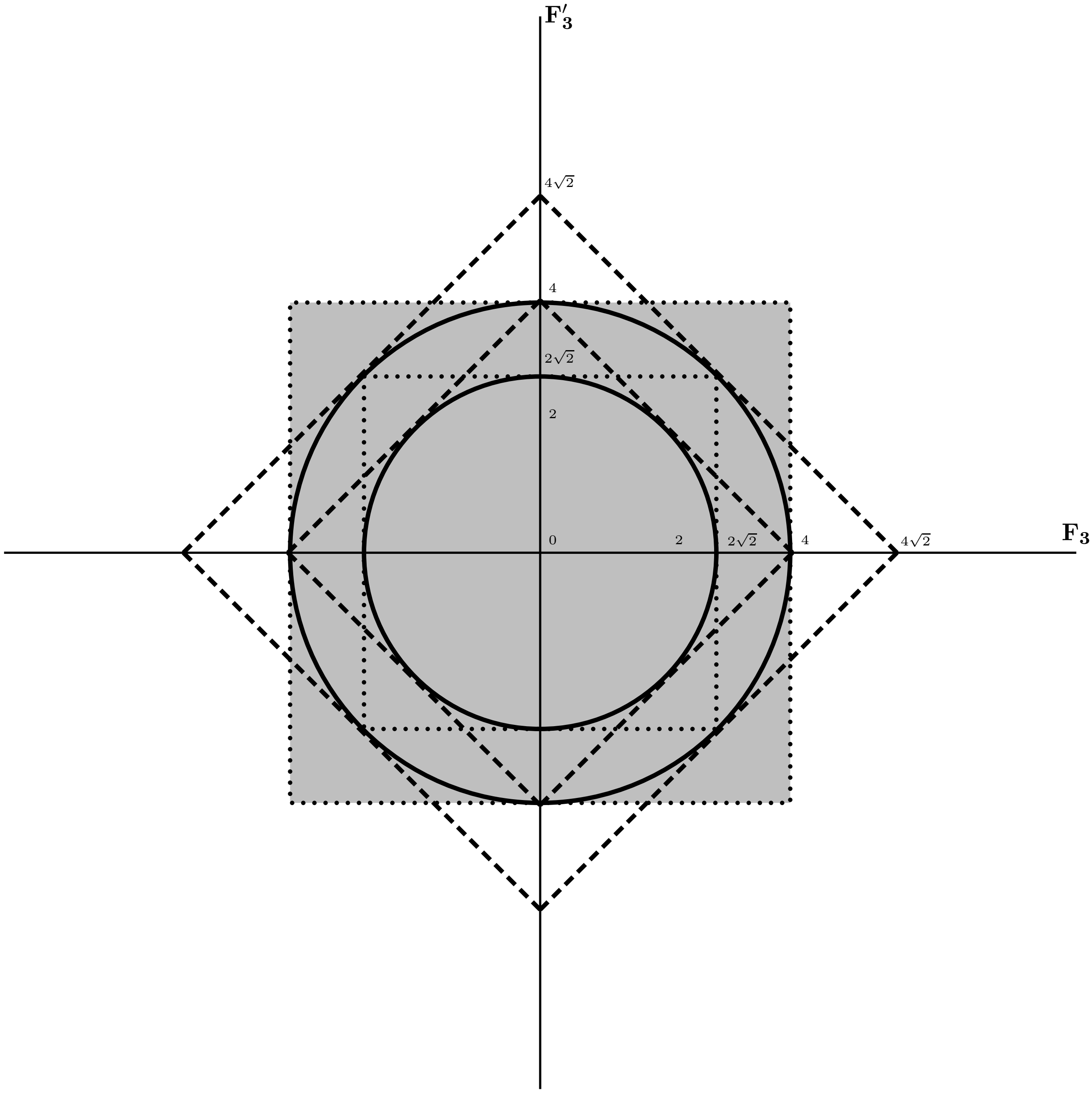
which is stronger than both the Svetlichny inequality and the Belinskii-Klyshko inequality. **It is the circle that just fits in their intersection.**

Further, for **all** ρ:

$$| < F_3 > | \leq 4, \quad | < F_3' > | \leq 4$$

$$| < S_3^{\pm} > | \leq 4\sqrt{2}$$

$$< F_3 >^2 + < F_3' >^2 \leq 16$$



$\mathbf{F_3'}$
$4\sqrt{2}$
4
$2\sqrt{2}$
2
0
2
$2\sqrt{2}$
4
$4\sqrt{2}$
$\mathbf{F_3}$

∞ Outlook ∞

- Revealing hidden entanglement via generalised measurements: From POV to POVM.

- What is the necessary and sufficient set of Bell-type inequalities that characterizes all possible partial factorisability?