Quantum Operations

and Measurement

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Second version

∞ Motivation ∞

... it took another 30 years or so [after John Bell's 1964 paper] for physicists to realize that entanglement was not so much a conceptual embarrassment for a respectable physical theory as a valuable resource that could be exploited to perform tasks impossible in the classical world. This realization has led to the emergence of a new field in quantum physics – or perhaps better, a new way of doing quantum physics – ...

Surprisingly, with few exceptions philosophers of physics have shown little interest in the relevance of these developments to the conceptual problems of quantum mechanics. In our view, the new work on quantum information changes the landscape of the old debate completely ...

As we see it, the locus of interesting work today on the foundational problems of quantum mechanics is the field of quantum information. This is were the seeds planted by Bohr and Einstein have finally taken root – the philosopher of science should be aware of the new garden.

Jeffrey Bub and Chris Fuchs. SHPMP, September 2003.

∞ Motivation ∞

- What are the most general dynamics and measurement operations quantum mechanics allows for?

  What do physical constraints, such as locality, imply on the allowed dynamics?

- I look at formalizing the system dynamics and measurement dynamics of open quantum systems.

  Only discrete dynamics, i.e. only final and initial states matter. Thus no continuous time-description such as using master equations.

∞ **Outline** ∞

1. The formalism of quantum operations.

2. Three ways of understanding the quantum operation.

3. Elementary operations.

4. Classification using physical constraints.

5. Measurement as a quantum operation.

6. Three examples

∞ The formalism of quantum operations ∞

Key ideas are two generalizations of the standard formalism:

- **Completely positive maps** generalize the unitary (free) evolutions of the standard quantum mechanics. These describe the most general evolution of open quantum systems.
- The **semi-spectral resolution of the identity** generalizes the spectral resolution of the identity.

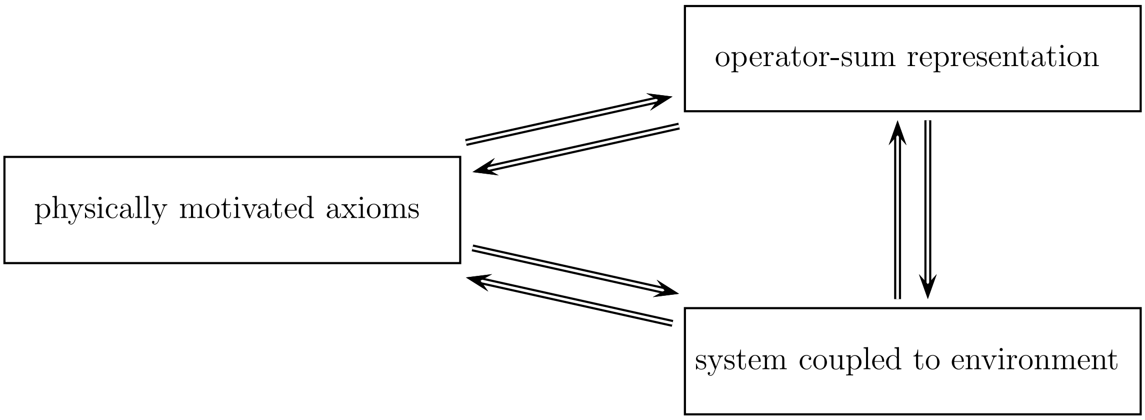
  $\sum_i M_i = \mathbb{1}$ generalizes $\sum_i P_i = \mathbb{1}$, with $P_i P_j = \delta_{ij} P_i$.

  The first is associated with a positive operator valued measure (POVM), the second with the well-known orthogonal resolution $A = \sum_i \lambda_i P_i$ of a self-adjoint operator A.

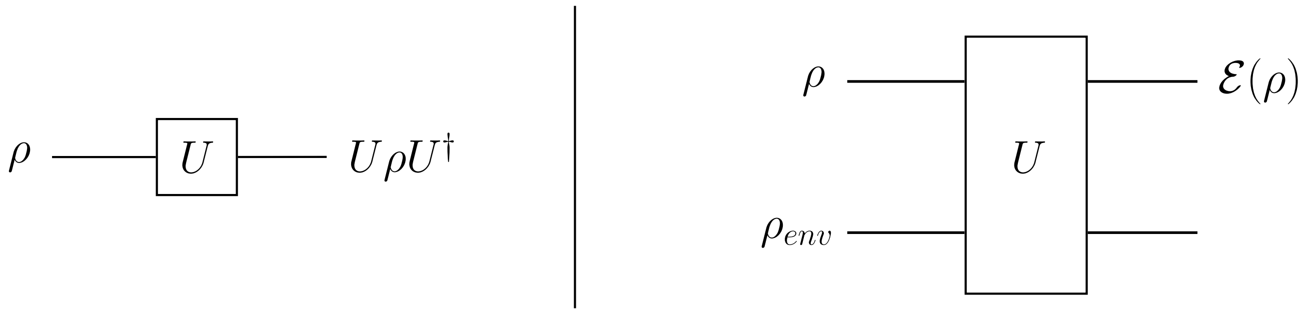
Quantum operation: Any physical process that takes a state ρ of a system on $\mathcal{H}_1$ to a state ρ' on $\mathcal{H}_2$. This process is described by a map $\mathcal{E} : \mathcal{H}_1 \to \mathcal{H}_2$,

$$\rho \to \rho' = \mathcal{E}(\rho).$$

There are three ways of understanding the quantum operation $\mathcal{E}$.



I: System coupled to environment



System dynamics arises from the free unitary evolution on a composed system. The principal system dynamics is:

$$\mathcal{E}(\rho) = \mathrm{Tr}_{env}\left[U(\rho \otimes \rho_{env})U^\dagger\right].$$

Furthermore, after the evolution it is possible to perform a projective von Neumann measurement.

This is also called the Stinespring dilation form of $\mathcal{E}$.

II: operator-sum representation

$$
\begin{aligned}
\mathcal{E}(\rho) &= \mathrm{Tr}_{env}\left[U(\rho \otimes \rho_{env})U^\dagger\right] \\
&= \sum_k E_k \rho_k E_k^\dagger \ ,
\end{aligned}
$$

E_k are the operation elements or Kraus elements and satisfy:

$$\sum_k E_k E_k^\dagger \leq \mathbb{1}.$$

This gives an **intrinsic means** of characterizing the dynamics of the principal system. There is no need to consider properties of the environment.

- **Physical interpretation** of operator sum representation:

The action of the quantum operation $\mathcal{E}(\rho)$ is equivalent to taking the state ρ and randomly replacing it by

$$E_k \rho_k E_k^\dagger / \mathrm{Tr}[E_k \rho_k E_k^\dagger] \ ,$$

with probability $\mathrm{Tr}[E_k \rho_k E_k^\dagger]$.

Note: The operator sum representation is not unique.

III: Physically motivated axioms

1. $\mathcal{E}(\rho)$ must act **linearly** on density matrices so that a mixture of input states leads to a mixture of output states.

2. $\mathcal{E}(\rho)$ must be **trace-nonincreasing**, because $\mathrm{Tr}[\mathcal{E}(\rho)]$ is defined as the probability that the process $\mathcal{E}(\rho)$ occurs.

3. $\mathcal{E}(\rho)$ must be **positive**, because since ρ is positive so must $\mathcal{E}(\rho)$ in order for it to be a density matrix.

4. $\mathcal{E}(\rho)$ must be **completely positive**. That is, $\mathcal{E}\otimes\mathbb{1}(\rho\otimes\sigma)$ must take density operators to density operators.

These four axioms **together imply** that the operation $\mathcal{E}(\rho)$ is a **completely positive trace non-increasing map**.

The three approaches to quantum operations are equivalent:

(A) $\mathcal{E}(\rho)$ is a completely positive trace non-increasing map **iff** it

(i) has a operator sum representation, or

(ii) comes from a unitary evolution and projective measurement on a larger system (Stinespring dilation).

(B) Any quantum operation can be physically realized:

(I) Any system - environment model gives rise to an operator sum representation:

$$\mathcal{E}(\rho) = \mathrm{Tr}_{env}\left[U(\rho \otimes \rho_{env})U^\dagger\right] = \sum_k E_k \rho E_k^\dagger,$$

with $E_k := \langle\, e_k |\, U \,|\, e_0 \rangle$, where $|\, e_0\rangle$ is the initial state of the environment.

(II) Conversely, a system-environment model can be given for any operator sum representation. Given a set of operation elements E_k, define the free evolution U of the composed system as:

$$U\,|\,\psi\rangle\,|\,e_0\rangle := \sum_k E_k\,|\,\psi\rangle\,|\,e_k\rangle\ .$$

Then the operator sum representation is realized by U:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \mathrm{Tr}_{env}[U(\rho \otimes |\, e_0\rangle\,\langle\, e_0|)U^\dagger]$$

Note: For trace-decreasing operations and extra projection onto the environment must be included.

∞ Elementary operations ∞

The following four **elementary operations** together imply a completely positive trace non-increasing map $\mathcal{E}(\rho)$:

(i) **Unitary transformations:**

$$\rho \longrightarrow \rho' = U \rho U^\dagger$$

(ii) **Von Neumann measurements:**

$$\rho \longrightarrow \rho' = \sum_i P_i \rho P_i$$

(iii) **Adding an uncorrelated system:**

$$\rho \longrightarrow \rho' = \rho \otimes \sigma$$

(iv) **Dismissal of part Q of a system:**

$$\rho \longrightarrow \rho' = \mathrm{Tr}_Q[\rho]$$

Theorem: A quantum operation is completely positive iff it can be composed out of the elementary operations (i) - (iv).

∞ Classification using constraints ∞

Definition: A **class of operations** is a set of operations that is closed under (i) composition, (ii) taking convex sums, (iii) taking tensor products and furthermore contains the identity.

Constraints on the operations give different classes. Then, using operational criteria that specify each class, one can determine if a quantum process meets the specific constraint.

- **LOCC operations** is the class of local operations plus two-way classical communication. It consists of composition of the following two elementary operations

$$\mathcal{E}^A \otimes \mathbb{1},$$
$$\mathbb{1} \otimes \mathcal{E}^B.$$

with $\mathcal{E}^A$ and $\mathcal{E}^B$ local quantum operations.

Example: A communicates her result α to B, after which B performs his measurement:

$$\mathcal{E}^{AB}(\rho) = (\mathbb{1} \otimes \mathcal{E}^B_\alpha) \circ (\mathcal{E}^A \otimes \mathbb{1})\rho$$

∞ Measurements ∞

Any measurement process can be described in terms of a quantum operation in the following way:

- To the set of possible outcomes $\{m\}$ from a measurement a set of quantum operations $\{\mathcal{E}_m\}$ is associated.

- Each $\mathcal{E}_m$ describes the dynamics of the system when outcome m is found.

- The probability p_m of the outcome m is $\mathrm{Tr}[\mathcal{E}_m(\rho)]$ and the post-measurement state is given by $\mathcal{E}_m(\rho)/\ \mathrm{Tr}[\mathcal{E}_m(\rho)$.

- The total quantum operation $\mathcal{E} = \sum_m \mathcal{E}_m$ is trace preserving, because the probabilities p_m the distinct outcomes sum to one.

Examples for measurement operation $\mathcal{E}_m = E_m \rho E_m^\dagger$:

(i) **von Neumann measurement:**

The operation element E_m is equal to the projector P_m.

(ii) **POVM measurement:** A set of operators $\{\mathcal{M}_m\}$ satisfying

$$\begin{array}{lll} (i) & M_m \geq 0 & \text{positivity} \\ (ii) & \sum_m M_m = \mathbb{1} & \text{completeness} \\ (iii) & p_m = \mathrm{Tr}[M_m \rho] & \text{probability rule} \end{array}$$

The POVM element M_m is equal to $E_m E_m^\dagger$ and the operation is then given by $\mathcal{E}_m(\rho) = \sqrt{M_m}\rho\sqrt{M_m}$.

Implementing the measurement.

A measurement model is possible for implementing the set of quantum operations $\mathcal{E}_m$ by performing a **unitary evolution** an a larger system and performing a **projective measurement** on the extra ancillary system.

This will lead to the correct system dynamics, outcome probability and post-measurement state.

∞ Examples ∞

(i) Can we create entanglement using only local means?
No, it is a central principle of quantum information theory that entanglement can not be created using LOCC.

But, can we nevertheless **measure entanglement** using only local means? To me this is still an open problem.

Bell measurement is claimed to be localizable using shared randomness. However, only the measurement dynamics on the system is implemented and no outcomes are obtained.

(ii) Entanglement as a catalyst.
What states can be locally obtained from some other state? The mere presence of entangled states allows for otherwise **impossible** local transformation to be realized, without the entanglement being consumed or altered. (Jonathan and Plenio, PRL 83, 3566 (1999))

(iii) Non-locality without entanglement.
Certain observables with only product eigen-states cannot be measured using LOCC. (Bennett et al. PRA 59, 1070 (1999))

∞ Conclusion ∞

- The new way of doing quantum physics uses the formalism of quantum operations. It allows for investigating what processes are possible by acting on a system under what specific constraints.

- Foundational questions can be formulated in this formalism so that an **operational investigation** of the problem becomes possible.

- We might get fundamental new insights from investigating what we **can** and **can not do** quantum mechanically.

  This idea gives rise to the so called **thermodynamical analogy**, because from investigating what we can and can not do thermodynamically we obtain the second law of thermodynamics.