Simulating Local Realism by Quantum Mechanics

Entanglement as a necessary resource to simulate

all local hidden variable correlations

∞ M.P. Seevinck ∞

E-mail: Seevinck@phys.uu.nl

Utrecht University, The Netherlands, April 2006.

∞ Motivation & Results ∞

Bell inequalities serve two purposes:

(A) Foundational: Quantum mechanics vs. Local hidden variable theories.

- Is a local realistic description of quantum mechanics possible?

- "Whereas in the late Eighties there was hardly any difference between entangled states and states violating a Bell inequality, we have a much more subtle discrimination nowadays."
  (R. Werner, 2001)

(B) Technical: Bell Inequalities give conditions for detecting entanglement (non-separability).

- Do we have **full** (N)-particle quantum entanglement and not just classical combinations of quantum entanglement of a smaller number of particles?

Quantum Mechanics vs. Local Hidden Variable Theories

- **Known**: The maximal violation of multipartite Bell inequalities obtainable by quantum mechanics grows exponentially for large N with respect to the value obtainable by LHV models.

Thus: Quantum mechanics is not about a local realistic structure build up out of values of physical quantities. Divergence, *not* convergence as the number of systems grows.

- **New Inequalities**: The maximum value obtainable by **separable** quantum states *exponentially decreases* with respect to the maximum value obtainable by a LHV model.

Thus: As the number of particles increases a larger and larger set of correlations that LHV models are able to give rise to *need entanglement* to be reproducible by quantum mechanics.

Why? It is precisely the quantum feature of *incompatible* (i.e. *complementary*) observables encoded via anticommutivity, which by itself is non-classical, that allows for the strange fact that a 'less than classical' feature arises in QM.

∞ Outline ∞

1. **Bi-partite ($N = 2$)**
   - Stronger CHSH-inequalities and quadratic inequalities for separable quantum states.
   - Local Hidden Variable consideration: entanglement is needed for a large set of LHV correlations.

2. **Multi-partite ($N = 3, \ldots, N$)**
   - Partial separability.
   - Mermin-type Bell inequalities for complementary observables.
   - Exponential divergence between correlations due to separable states and LHV correlations.

3. **Discussion**
   - Quantum and classical correlations revisited.

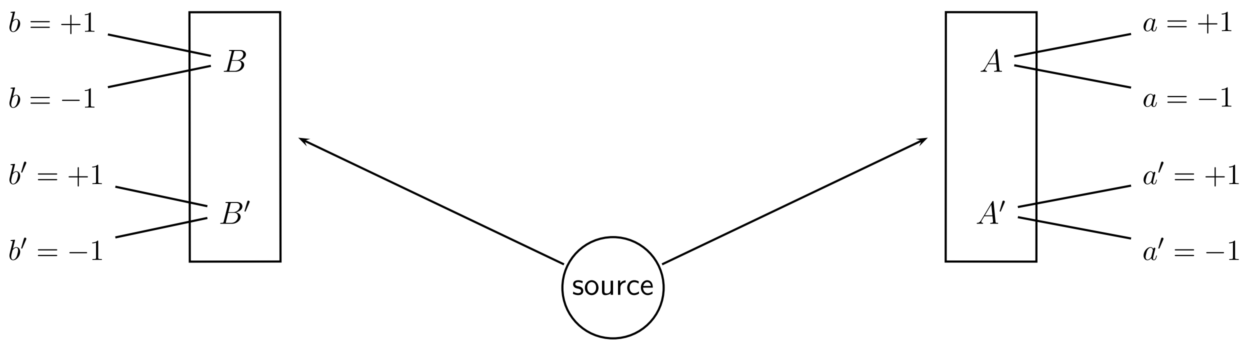
∞ **Bi-partite case ($N = 2$)** ∞

(i) Two spin-$\frac{1}{2}$ particles: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

(ii) A, A' are spin observables on the first particle; B, B' on the second (dichotomic (± 1-valued).

(iii) The expectation value of $A \otimes B$ in the mixed or pure state ρ is $\langle AB \rangle_\rho := \text{Tr}\,[\rho A \otimes B]$.

Paradigmatic setup:



• The CHSH-inequality says that for non-entangled states, i.e., for states of the form $\rho = \rho_1 \otimes \rho_2$, or mixtures of such states:

$$|\langle AB + AB' + A'B - A'B' \rangle_\rho| \leq 2. \tag{1}$$

• The maximal violation of (1) for entangled states follows the Cirel'son bound:

$$|\langle AB + AB' + A'B - A'B' \rangle_\rho| \leq 2\sqrt{2}. \tag{2}$$

Equality can be attained by the Bell states: $\frac{1}{\sqrt{2}}|\uparrow\uparrow \pm \downarrow\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\downarrow \pm \downarrow\uparrow\rangle$.

• Uffink's quadratic inequality shows that for all states ρ

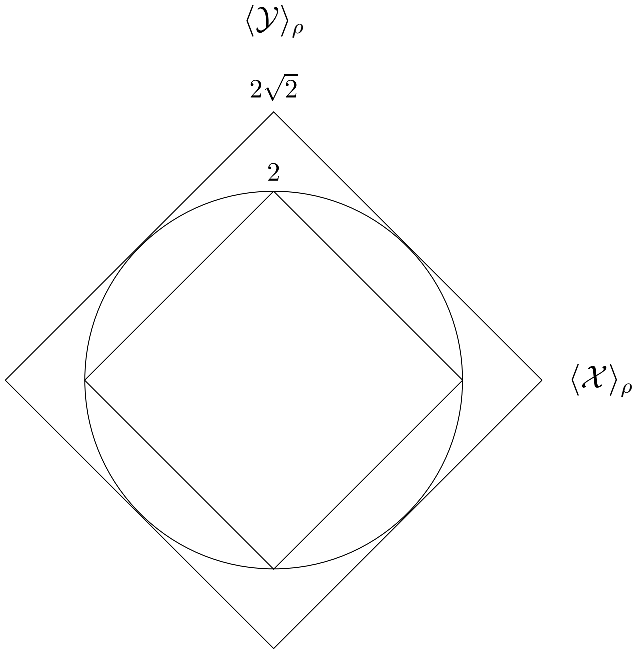
$$\langle AB' + A'B \rangle_\rho^2 + \langle AB - A'B' \rangle_\rho^2 \leq 4, \tag{3}$$

which strengthens the Cirel'son inequality (2).

• Unfortunately, **no** smaller bound on the left-hand side of the quadratic inequality (3) exists for non-entangled states, as long as the choice of observables is kept *completely general.*

To verify this, take $|\psi\rangle = |\uparrow\uparrow\rangle$ and $A = A' = B = B' = \sigma_z$.

• Thus, the quadratic inequality (3) does **not** distinguish entangled and non-entangled states.



The region of allowed values in the $(\mathcal{X}, \mathcal{Y})$-plane, where $\mathcal{X} = \langle AB' + A'B \rangle_\rho$, and $\mathcal{Y} = \langle AB - A'B' \rangle_\rho$.

∞ Strengthening the inequalities for separable states ∞

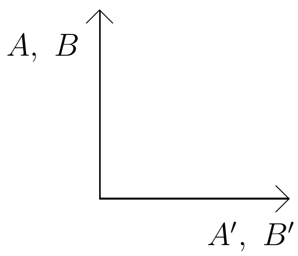
A much more stringent bound can be found for non-entangled states when a suitable choice of observables is made.

Observation: The only choice of observables that gives maximal violations of the CHSH inequality (1) is locally choosing **complementary observables**.

That is, choose A, A' as spin-components in mutually orthogonal directions, and similarly for B, B'.

$\Longrightarrow$ anti-commutativity: $\{A, A'\} = \{B, B'\} = \mathbf{0}$.

Example:
The choice $A = \sigma_x,\ A' = \sigma_y$ and $B = \sigma_x,\ B' = \sigma_y$ gives the maximum violation for $|\,\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{2}(1+i)\,|\downarrow\downarrow\rangle$.



Strengthening the inequalities for separable states:

(**1**) Let us make such a choice: $\{A, A'\} = \{B, B'\} = \mathbf{0}$.

(**2**) Assume that the two-particle state is pure and non-entangled.

$\Longrightarrow$ We may thus write $\rho = |\,\Psi\rangle\,\langle\,\Psi|$, where $|\,\Psi\rangle\ = |\,\psi\rangle\,|\,\phi\rangle$, and obtain **factorisability**: $\langle AB\rangle_\rho = \langle A\rangle\langle B\rangle$.

This factorisability gives

$$\begin{aligned}&\langle AB' + A'B\rangle_\rho^2 + \langle AB - A'B'\rangle_\rho^2 =\\&\left(\langle A\rangle\langle B'\rangle + \langle A'\rangle\langle B\rangle\right)^2 + \left(\langle A\rangle\langle B\rangle - \langle A'\rangle\langle B'\rangle\right)^2 =\\&\left(\langle A\rangle^2 + \langle A'\rangle^2\right)\left(\langle B\rangle^2 + \langle B'\rangle^2\right).\end{aligned}$$

(**3**) Note that for any pure spin-$\frac{1}{2}$ state, and any choice of an **orthogonal** triple of spin components

$$\langle A\rangle^2 + \langle A'\rangle^2 + \langle A''\rangle^2 = 1.$$

This gives:

$$\begin{aligned}(\langle A\rangle^2 + \langle A'\rangle^2 &= 1 - \langle A''\rangle^2\\(\langle B\rangle^2 + \langle B'\rangle^2) &= 1 - \langle B''\rangle^2.\end{aligned}$$

Thus we obtain

$$\langle AB' + A'B\rangle_\rho^2 + \langle AB - A'B'\rangle_\rho^2 \leq \left(1 - \langle A''\rangle^2\right)\left(1 - \langle B''\rangle^2\right) \leq 1.$$

$$\iff \quad \langle AB' + A'B \rangle_\rho^2 + \langle AB - A'B' \rangle_\rho^2 \leq 1. \qquad (4)$$

$\Longrightarrow$ As a result, the quadratic inequality is now bounded by 1 for any pure product state.

The result can be extended to **any non-entangled mixed state** by noting that **(i)** a mixed state is a convex combination of the density operators for pure product-states, i.e. $\rho = \sum_i p_i \,|\, \psi_i\rangle \langle\, \psi_i|$ and **(ii)** that the left hand side of (4) is a convex function of ρ.

- In fact, **any** pure state that satisfies this quadratic inequality is separable.

$\Longrightarrow$ Thus, we have found a *sufficient and necessary* condition for separability of pure states of two spin-$\frac{1}{2}$ particles.

Corrolary: The strengthened CHSH inequality for separable (i.e. non-entangled) states becomes

$$|\langle AB + AB' + A'B - A'B' \rangle_\rho| \leq \sqrt{2} \qquad (5)$$

- The above bounds do not hold for entangled states.

For example, for the singlet $|\,\Psi\rangle = \frac{1}{\sqrt{2}}\,|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$ one finds:

(I) The choice

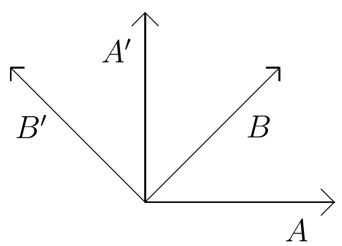
$$\langle\sigma_x \otimes \sigma_x\rangle = 1,\ \langle\sigma_y \otimes \sigma_y\rangle = -1,$$
$$\langle\sigma_x \otimes \sigma_y\rangle = \langle\sigma_y \otimes \sigma_x\rangle = 0,$$

gives:

$$\langle AB' + A'B\rangle_\rho^2 + \langle AB - A'B'\rangle_\rho^2 = 4 \not\leq 1.$$

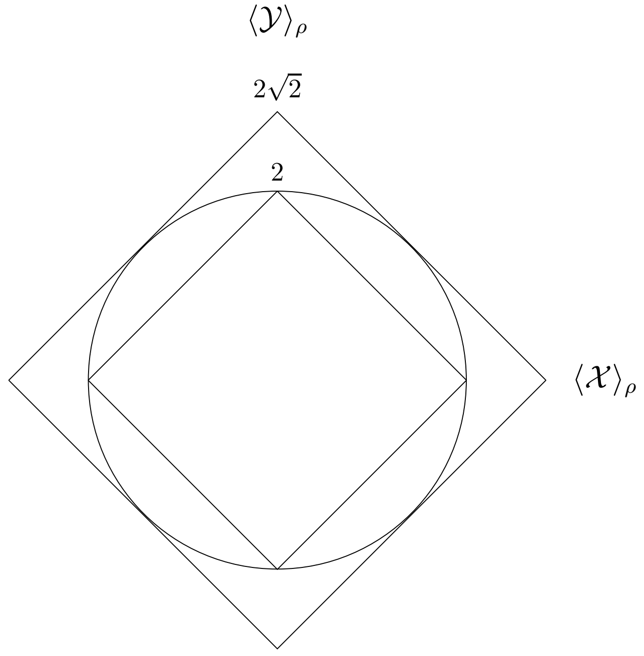
(II) The choice

$$A = \sigma_x,\ A' = \sigma_y$$
$$B = \sigma_x + \sigma_y,\ B' = \sigma_y - \sigma_x,$$

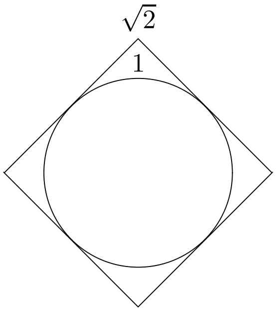


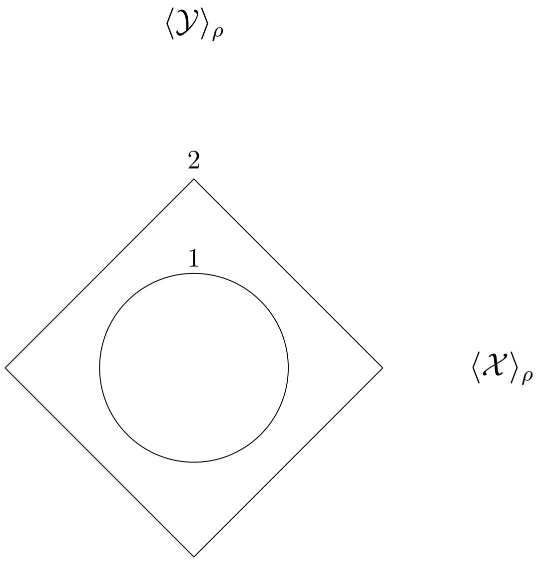
gives:

$$|\langle AB + AB' + A'B - A'B'\rangle_\rho| = 2\sqrt{2} \not\leq \sqrt{2}.$$



The region of allowed values in the $(\mathcal{X},\mathcal{Y})$-plane, where $\mathcal{X} = \langle AB' + A'B\rangle_\rho$, and $\mathcal{Y} = \langle AB - A'B'\rangle_\rho$.


$\sqrt{2}$
1



The well-known CHSH inequality compared to the strengthened quadratic inequality for separable states.

$$\begin{aligned} CHSH &= AB + A'B + AB' - A'B', \\ (CHSH)' &= A'B' + AB' + A'B - AB. \end{aligned}$$

(1) Then for general A, A' and B, B' we know (Cirel'son):

$$\max |\langle CHSH \rangle_\rho|, |\langle (CHSH)' \rangle_\rho| \leq 2\sqrt{2}. \tag{6}$$

(2) And for separable states (again for general A, A' and B, B'):

$$\max |\langle CHSH \rangle_\rho|, |\langle (CHSH)' \rangle_\rho| \leq 2. \tag{7}$$

(3) We also know the quadratic inequality:

$$\langle CHSH \rangle_\rho^2 + \langle (CHSH)' \rangle_\rho^2 \leq 8. \tag{8}$$

This follows from

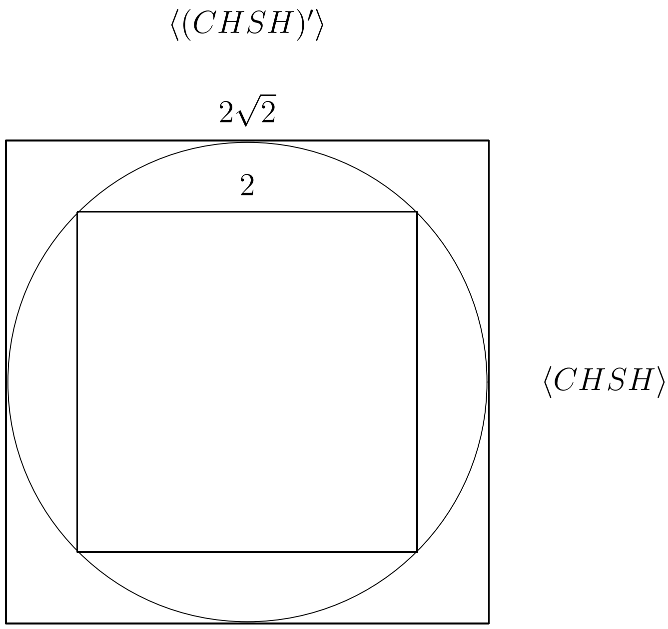
$$\langle CHSH \rangle_\rho^2 + \langle (CHSH)' \rangle_\rho^2 = 2\,(\langle A'B + AB' \rangle_\rho^2 + \langle AB - A'B' \rangle_\rho^2 \leq 2 \cdot 4 = 8.$$

- However, for orthogonal measurements we get the **sharper** quadratic inequality:

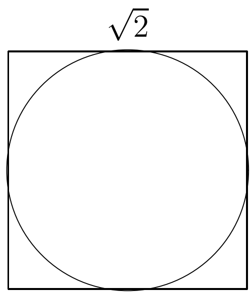
$$\langle CHSH \rangle_\rho^2 + \langle (CHSH)' \rangle_\rho^2 \leq 2. \tag{9}$$

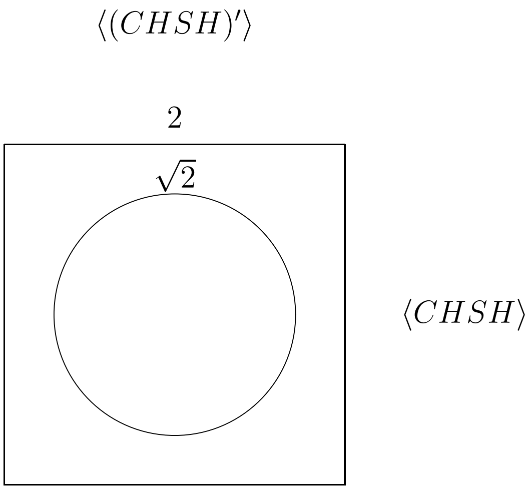
From this we get the linear inequalities:

$$\max |\langle CHSH \rangle_\rho|, |\langle (CHSH)' \rangle_\rho| \leq \sqrt{2}. \tag{10}$$



$\langle (CHSH)' \rangle$
$2\sqrt{2}$
2
$\langle CHSH \rangle$


$\sqrt{2}$



The well-known CHSH inequality compared to strengthened quadratic inequality for separable states.

∞ Comparison to local hidden variable theories ∞

- Do we get a similar strengthening in the context of local hidden variable theories?

It is well known that the CHSH inequality

$$|\langle AB + AB' + A'B - A'B' \rangle_\lambda| \leq 2, \tag{11}$$

holds for any theory in which dichotomous outcomes $a, b \in \{+, -\}$ are subjected to a factorisable probability distribution

$$p(a, b) = \int d\lambda \rho(\lambda) P_{\vec{a}}(a|\lambda) P_{\vec{b}}(b|\lambda) \tag{12}$$

where $\vec{a}, \vec{b}$ denote the 'parameter settings', i.e. the directions of the spin components measured.

- The assumption to be **added** to such a theory in order to obtain the strengthened inequality

$$|\langle AB + AB' + A'B - A'B' \rangle_\lambda| \leq \sqrt{2}, \tag{13}$$

is the requirement that for any orthogonal choice of $\vec{a}$ and $\vec{a'}$ we have

$$\langle A \rangle_\lambda^2 + \langle A' \rangle_\lambda^2 \leq 1, \tag{14}$$

where $\langle A \rangle_\lambda = \sum_{a=\pm} a\, p_{\vec{a}}(a|\lambda)$, etc.

- But this requirement is *by no means obvious* for the local hidden variable theory. After all, as has often been pointed out, such a theory may employ a mathematical framework which is completely different from quantum theory (e.g. Bell's critique on von Neumann's additivity assumption in his 'no-go' proof).

There is no *a priori* reason why the orthogonality of spin directions should have any particular significance in the hidden variables theory, and the why it should confirm to quantum mechanics in reproducing (14).

Indeed, it is violated by Bell's own example of a LHV model. Thus, the additional requirement (14) would be unmotivated within a LHV theory.

Thus: Testing for entanglement within quantum theory and testing quantum mechanics against LHV theories are **not** equivalent.

- Of course, this conclusion is not new: Werner has already shown that an explicit LHV theory can be constructed for a family of entangled states , the so-called Werner states:

$$\rho_W = \frac{1}{8}\mathbb{1} + \frac{1}{4} |\, \Psi_s \rangle \, \langle \, \Psi_s | \, .$$

- However, the phenomenon exhibited by Werner states is much more ubiquitous, and many more mixed entangled states obey Bell inequalities and have a LHV model.

Proof:

Let $\alpha, \beta, \alpha', \beta'$ denote spin observables in *arbitrary* directions. We can find associated orthogonal directions by the transformations

$$2\cos\phi A = (\alpha + \alpha'), \quad 2\sin\phi A' = \alpha - \alpha',$$
$$2\cos\psi B = (\beta + \beta'), \quad 2\sin\psi B' = \beta - \beta'.$$

Hence, the Bell-CHSH expression is transformed into

$$\langle \alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' \rangle \; = \;$$
$$2\left(\cos\phi\cos\psi\langle AB\rangle + \sin\phi\cos\psi\langle AB'\rangle\right.$$
$$\left. + \cos\phi\sin\psi\langle A'B\rangle - \sin\phi\sin\psi\langle A'B'\rangle\right)$$

Take, for example, some state ρ such that

$$\langle AB\rangle_\rho \; = \; -\langle A'B'\rangle_\rho \; =: x, \quad \langle AB'\rangle_\rho \; = \; \langle A'B\rangle_\rho \; =: y$$

then, according to the quadratic inequality, ρ is entangled whenever $x^2 + y^2 > \frac{1}{4}$. But since

$$\langle \alpha\beta + \alpha\beta' + \alpha'\beta - \alpha'\beta' \rangle_\rho \; =$$
$$2\left((\cos(\phi - \psi)x + (\sin(\phi + \psi)y\right) \; \leq \; 2(x + y),$$

such a state ρ will satisfy Bell inequalities for all observables when $x + y \leq 1$.

Hence, every state for which both $x^2 + y^2 > \frac{1}{4}$ and $x + y \leq 1$ is true (e.g. $\frac{1}{2\sqrt{2}} \leq x = y < \frac{1}{2}$) will be be both entangled and allow a LHV model for every pair of observables.

∞ Conclusions obtained so far ∞

- The strengthened quadratic Bell inequality does not hold in LHV theories. This provides a more general example of the fact first discovered by Werner, i.e. that some entangled states do allow a LHV reconstruction for all observables.

- There exists a 'gap' between the correlations that can be obtained by non-entangled quantum states and those by LHV models. This means that, apart from the question raised and answered by Bell (can the predictions of quantum mechanics be reproduced by a LHV theory?) it is *also* interesting to ask whether non-entangled quantum states can reproduce the predictions of a LHV theory. The answer, as we have seen, is **negative**:

Quantum theory generally needs entangled states to reproduce the classical correlations of such a LHV theory.

- In fact the gap between the correlations allowed by local hidden variable theories and those achievable by non-entangled quantum states **increases exponentially** with the number of particles.

$\Longrightarrow$ **Multi-particle generalisation.**

Key idea for generalisation

- Note the following operator identity for the choice

$$\begin{aligned} \mathcal{X} &:= \sigma_x^1\sigma_y^2 + \sigma_y^1\sigma_x^2 = -2i\,(\,|\uparrow\uparrow\rangle\,\langle\downarrow\downarrow| - |\downarrow\downarrow\rangle\,\langle\uparrow\uparrow|\,), \\ \mathcal{Y} &:= \sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2 = 2\,(\,|\uparrow\uparrow\rangle\,\langle\downarrow\downarrow| + |\downarrow\downarrow\rangle\,\langle\uparrow\uparrow|\,), \end{aligned} \qquad (15)$$

- Then $\langle\mathcal{X}\rangle_\rho = 4\mathrm{Im}\,\rho^{(2)}_{\nearrow}$ and $\langle\mathcal{Y}\rangle_\rho = 4\mathrm{Re}\,\rho^{(2)}_{\nearrow}$,

(where $\rho^{(2)}_{\nearrow} = \langle\,\uparrow\uparrow|\,\rho\,|\downarrow\downarrow\rangle$ is the anti-diagonal matrix element).

$$\begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x^* & 0 & 0 & 0 \end{pmatrix}$$

- The strengthened quadratic inequality can thus also be formulated as stating that for all non-entangled states:

$$\langle\mathcal{X}\rangle_\rho^2 + \langle\mathcal{Y}\rangle_\rho^2 = 16|\rho^{(2)}_{\nearrow}|^2 \leq 1, \quad \Longleftrightarrow \quad |\rho^{(2)}_{\nearrow}| \leq \frac{1}{4}. \qquad (16)$$

- And the strengthened Bell-inequality can be rederived:

$$\begin{aligned} |\langle\sigma_x^1\sigma_x^2 + \sigma_x^1\sigma_y^2 + \sigma_y^1\sigma_x^2 - \sigma_y^1\sigma_y^2\rangle_\rho| &= 4\mathrm{Im}\,\rho^{(2)}_{\nearrow} + 4\mathrm{Re}\,\rho^{(2)}_{\nearrow} \\ &\leq \sqrt{2}. \end{aligned} \qquad (17)$$

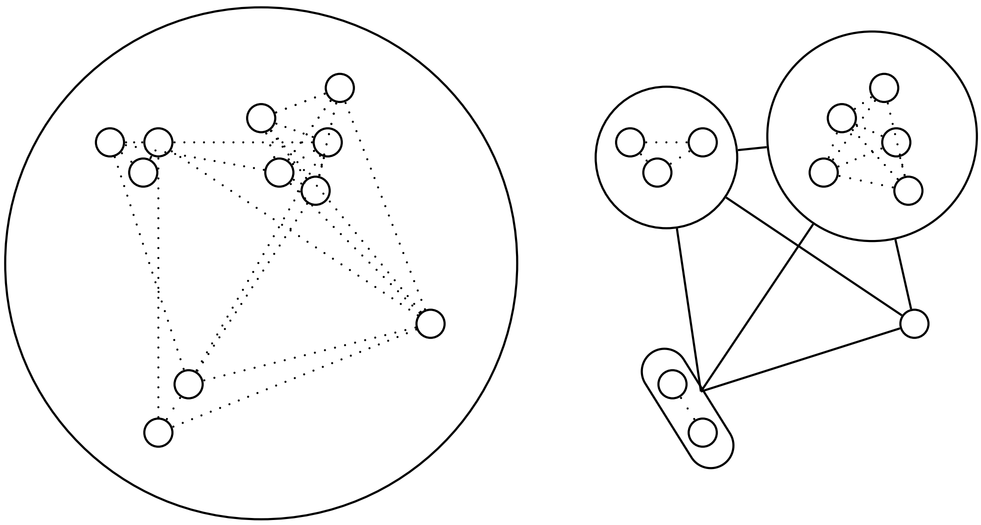
∞ Step (1): Partial Separability ∞

- Consider an N-particle system and let $S_1, \ldots, S_k$ denote a partition of $\{1, \ldots, N\}$ into k disjoint and nonempty subsets.

- A quantum state $\rho_N^{k-\text{sep}}$ of this N-particle system is called k-separable ($k \leq N$) iff there exists a convex decomposition

$$\rho_N^{k-\text{sep}} = \sum_i p_i \otimes_{n=1}^k \rho^{S_n^{(i)}} \qquad p_i \geq 0, \quad \sum_i p_i = 1, \qquad (18)$$

where each density matrix $\otimes_{n=1}^k \rho^{S_n^{(i)}}$ is a tensor product of k density matrices of the subsystems corresponding to some such partition.



∞ Step (2): 'At most k-separable' ∞

- A state that is k-separable but not $(k+1)$-separable we call 'at most k-separable'.

Clearly, for $k < N$, an N-particle state that is at most k-separable is not fully separable and therefore entangled.

- To emphasize this point these states are also refered to as 'k-separable entangled'.

∞ Step (3): Partial separability condition ∞

A very usefull condition for partial separability, first mentioned by Laskowksi and Żukowski , is provided by the fact that the modulus of any antidiagonal element of a k-separable density matrix satisfies

$$|\rho^{k-\text{sep}}_{\text{antidiagonal}}| \leq \left(\frac{1}{2}\right)^k, \qquad \forall \rho^{k-\text{sep}}_{\text{antidiagonal}} \,. \tag{19}$$

This condition, which will do must of the work still to come, can be easily proven by the observation that for any density matrix to be physically meaningfull we have that the antidiagonal matrix element must be less than 1/2.

- This condition also gives a condition for a 'at most k-separable' state or a 'k-separable entangled' state.

∞ Step (4): Generating multipartite Bell inequalities ∞

- The conditions for partial separability are in terms of the anti-diagonal matrix elements, and are thus not directly experimentally accessible.

Therefore, they are turned into multipartite Bell-type inequalities.

∞ **Example:** $N = 3$ ∞

- Let us consider the so-called Mermin operator for three qubits:

$$M_3 := \sigma_x^1\sigma_x^2\sigma_y^3 + \sigma_y^1\sigma_x^2\sigma_x^3 + \sigma_x^1\sigma_y^2\sigma_x^3 - \sigma_y^1\sigma_y^2\sigma_y^3. \tag{20}$$

Again only locally orthogonal observables are considered.
No loss of generality.

- Now note the following operator identity:

$$M_3 = -4i\,(|\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow| - |\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow|)\,, \tag{21}$$

so that for all states ρ

$$\langle M_3\rangle_\rho = 8\,\mathrm{Im}\,\langle\uparrow\uparrow\uparrow|\,\rho\,|\downarrow\downarrow\downarrow\rangle = 8\,\mathrm{Im}\,\rho^{(3)}_{\nearrow}, \tag{22}$$

where $\rho^{(3)}_{\nearrow} = \langle\uparrow\uparrow\uparrow|\,\rho\,|\downarrow\downarrow\downarrow\rangle$.

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Because $\mathrm{Im}\,\rho^{(3)}_{\nearrow} \leq |\rho^{(3)}_{\nearrow}|$ we can apply the partial separability criterion to get the following Mermin-type inequalities:

$$
\begin{aligned}
|\langle M_3\rangle_\rho| &\leq 1 \qquad \text{full separability } (k=3),\\
|\langle M_3\rangle_\rho| &\leq 2 \qquad \text{bi-separability } (k=2),\\
|\langle M_3\rangle_\rho| &\leq 4 \qquad \text{full entanglement } (k=1).
\end{aligned}
$$

Quadratic inequality for $N = 3$:

- We use the operator identity for M_3' which is same as M_3 but with all σ_x and σ_y interchanged:

$$\begin{aligned} M_3' : &= \sigma_y^1 \sigma_y^2 \sigma_x^3 + \sigma_y^1 \sigma_x^2 \sigma_y^3 + \sigma_y^1 \sigma_x^2 \sigma_y^3 - \sigma_x^1 \sigma_x^2 \sigma_x^3 \\ &= -4 \left(|\uparrow\uparrow\uparrow\rangle \langle \downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\rangle \langle \uparrow\uparrow\uparrow| \right). \end{aligned}$$

- The expectation value gives:

$$\langle M_3' \rangle_\rho = -8 \, \mathrm{Re} \, \langle \uparrow\uparrow\uparrow | \, \rho \, | \downarrow\downarrow\downarrow \rangle = -8 \mathrm{Re} \, \rho_{\nearrow}^{(3)}.$$

- Using the separability criterion of Eq. (19) we get the quadratic inequality for k-separability:

$$\begin{aligned} \langle M_3 \rangle_\rho^2 + \langle M_3' \rangle_\rho^2 &= 64 \Big((\mathrm{Re} \, \rho_{\nearrow}^{(3)})^2 + (\mathrm{Im} \, \rho_{\nearrow}^{(3)})^2 \Big) \\ &= 64 |\rho_{\nearrow}^{(3)}|^2 \le 64 \Big(\frac{1}{4} \Big)^k. \end{aligned} \tag{23}$$

- For full entanglement, bi-separable entanglement and full separability we find on the right hand side of Eq. (23) respectively the values 16, 4 and 1.

- These quadratic inequalities thus give sufficient conditions for all forms of k-separability in the three particle system.

- These are even stronger than the linear Bell-type inequalities since the latter are implied by the quadratic inequalities, which thus detect less entangled states.

∞ **Multipartite generalisation** ∞

- The Mermin operator for general N is defined via the following recursive chain:

$$M_N := \frac{1}{2} M_{N-1}(\sigma_x^N + \sigma_y^N) + \frac{1}{2} M'_{N-1}(\sigma_x^N - \sigma_y^N). \tag{24}$$

- The maximal value of $|\langle M_N \rangle_\rho|$ for any quantum state ρ is denoted as $M_N^{\max}$ which is known to be equal to $2^{(N+1)/2}$.

- This maximum is obtained by the N particle GHZ states $\left| \psi_{\mathrm{GHZ}}^N \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \uparrow^{\otimes N} \right\rangle + \left| \downarrow^{\otimes N} \right\rangle \right)$ for the case of orthogonal observables only.

- Note that local realism (i.e., LHV models) gives the maximal value for the expectation value of M_N equal to 2, which is a factor $2^{(N-1)/2}$ smaller than the quantum maximum.

New inequality: If

$$|\langle M_N \rangle_\rho| > M_N^{\max} \left(\frac{1}{2}\right)^k = 2^{\left(\frac{N+1}{2}\right)} \left(\frac{1}{2}\right)^k, \tag{25}$$

then the N particle state ρ is not $(k+1)$-separable but it is at most k-separable.

Proof of Eq. (25):

- Consider the general operator identity (in the z_1-z_2-...-z_N product basis):

$$M_N = 2^{(N+1)/2}\big(e^{i\alpha_N}\,|\downarrow^{\otimes N}\rangle\langle\uparrow^{\otimes N}| + e^{-i\alpha_N}\,|\uparrow^{\otimes N}\rangle\langle\downarrow^{\otimes N}|\big), \tag{26}$$

where $\alpha_N := \frac{\pi}{4}(N-1)$.

- Using the operator identity for M_N and the partial separability criterion the maximal value of $|\langle M_N\rangle|$ by a k-separable quantum state is then obtained as follows:

$$\begin{aligned} M^{\max}_{N,k-\mathrm{sep}} &= \sup_{\rho^{k-\mathrm{sep}}} |\langle M_N\rangle_\rho| = \sup_{\rho^{k-\mathrm{sep}}} 2|\rho^{(N)}_{\nearrow}|\,2^{(\frac{N+1}{2})} \\ &= 2\Big(\frac{1}{2}\Big)^k\,2^{(\frac{N+1}{2})} = 2^{(\frac{N-2k+3}{2})}, \end{aligned} \tag{27}$$

where $\rho^{(N)}_{\nearrow}$ is the antidiagonal matrix element $\langle\uparrow^{\otimes N}|\,\rho\,|\downarrow^{\otimes N}\rangle$. Thus for a k-separable state we have $|\langle M_N\rangle_\rho| < M^{\max}_{N,k-\mathrm{sep}}$.

- Thus if $|\langle M_N\rangle_\rho| > M^{\max}_{N,(k+1)-\mathrm{sep}}$ then the state ρ is not $(k+1)$-separable and at most k-separable. ■

• We get for the special case $k = N$ the full separability inequality

$$\langle M_N \rangle_\rho \leq 2^{(3-N)/2} \tag{28}$$

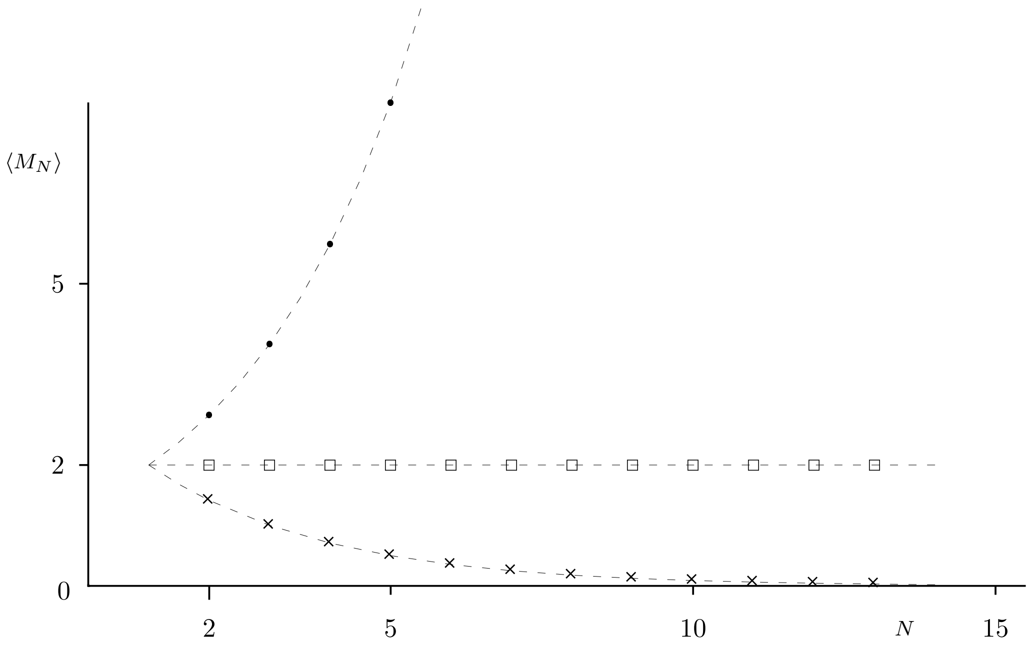
• This inequality is maximally violated by fully entangled states by an exponentially large factor of 2^{N-1}, whereas local realism (i.e., LHV modelds) violates this with the also exponentially large factor of $2^{(N-1)/2}$ (i.e., local realism gives $\langle M_N \rangle_\rho \leq 2$).

$\Longrightarrow$ exponential divergence.

• That the maximum violation of multipartite Bell inequalities obtainable by quantum mechanics grows exponentially for large N with respect to the value obtainable by LHV models has been know for quite some years.

• However, new is that the maximum value obtainable by separable quantum states *exponentially decreases* with respect to the maximum value obtainable by LHV models.

Thus as the number of particles increases a larger and larger set of correlations that LHV models are able to give rise to need entanglement to be reproducible by quantum mechanics.


⟨M_N⟩
5
2
0
2
5
10
N
15

Quadratic inequalities:

- The linear inequalities can be strengthened by the quadratic inequalities for k-separability. These follow from the operator identities for M_N in the following way:

$$\begin{aligned} \langle M_N\rangle_\rho^2 + \langle M'_N\rangle_\rho^2 &= 2^{(N+1)}\Big(4(\mathrm{Re}\,\rho_{\nearrow}^{(N)})^2 + 4(\mathrm{Im}\,\rho_{\nearrow}^{(N)})^2\Big) \\ &= 2^{(N+3)}|\rho_{\nearrow}^{(N)}|^2 \leq 2^{(N+3)}\big(\frac{1}{4}\big)^k. \end{aligned} \tag{29}$$

- Note that k and $(k+1)$-separability differ a factor of four between the values they can obtain for the left hand side of Eq. (29).

∞ Conclusion ∞

Back to the title of the talk:

Simulating Local Realism by Quantum Mechanics: Entanglement as a necessary resource to simulate all local hidden variable correlations.

- Usually Bell inequalities are interesting only if there exist a certain quantum state that violates it. However, we here see that it is also very interesting to ask not what quantum states violate a certain Bell inequality, but what quantum states **cannot** violate such a Bell inequality, and by what factor.

- The maximum value of multipartite Bell inequalities obtainable by separable quantum states **exponentially decreases** with respect to the maximum value obtainable by LHV models. Thus as the number of particles increases a larger and larger set of correlations that LHV models are able to give rise to need entanglement to be reproducible by quantum mechanics.

- It is precisely the quantum feature of **incompatible** (i.e. complementary) observables encoded via anticommutivity, which by itself is non-classical, that allows for the strange fact that a '*less than classical*' feature arises in QM.