Deep Hidden Variables

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Prospects & Introduction

I will reconsider the well-known (local) hidden variable program and the famous CHSH inequality.

Some **elementary** investigations and new results are presented that I believe to have general repercussions.

These are intended to deepen our understanding of what it takes to violate the Bell inequality and how this relates to quantum and no-signaling correlations.

As part of the recent 'paradigm change' to study quantum mechanics (QM) 'from the outside, not just from the inside'.

Study QM ‘from the outside, not just from the inside’

Motto: In order to understand quantum mechanics it is useful to demarcate those phenomena that are essentially quantum, from those that are more generically non-classical.

Investigate theories that are neither classical nor quantum; explore the space of possible theories from a larger theoretical point of view.

> *“Is quantum mechanics an island in theory space?”* (Aaronson, 2004). If indeed so, where is it?

- It is found that many non-classical properties of QM are generic within the larger family of physical theories.

Thus rather than regard quantum theory special for having the generic quantum properties, a better attitude may be to regard classical theories as special for not having them.

Methodological morale for this talk:

> *Now it is precisely in cleaning up intuitive ideas for mathematics that one is likely to throw out the baby with the bathwater.*
>
> *J.S. Bell; 'La nouvelle cuisine', 1990.*

► So I will keep things simple, and use a minimal of mathematics.

But the well-known simplest case, the setup of the EPR-Bohm experiments – already studied for over 40 years –, is not so simple after all: there is still very much to be discovered.

Outline

(I) Review of (local) hidden-variable models for the EPR-Bohm setup

(II) CHSH inequality revisited

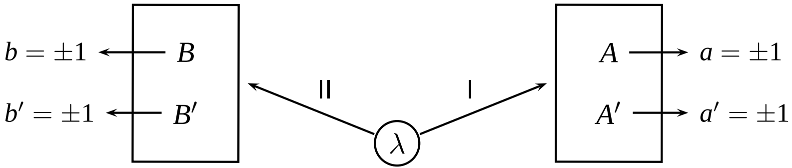
(III) Beyond LHV and QM

- Surface vs. subsurface level
- From non-locality to no-signaling

(VI) Conclusion and discussion

Local realism and hidden variables

Setup of the (2,2,2) *Gedankenexperiment*:



1. One assumes that the particle pair and other relevant degrees of freedom are captured in some physical state $\lambda \in \Lambda$ ('beables').

2. The model gives the probability for obtaining outcomes a, b when measuring A, B on a system in the state λ: $\quad P(a, b|A, B, \lambda)$.

3. Empirically accessible probabilities of outcomes are obtained by averaging over some probability density on λ:

$$P(a, b|A, B) = \int_\Lambda P(a, b|A, B, \lambda)\rho(\lambda|A, B)d\lambda.$$

Conditions imposed on the model

– Factorisability (Bell called this ‘local causality’):

$$P(a,b|A,B,\lambda) = P(a|A,\lambda)P(b|B,\lambda).$$

– Independence of the Source (IS): $\rho(\lambda|A,B) = \rho(\lambda)$.

- **Consequences of the assumptions:**

Factorisability $\wedge$ IS $\implies P(a,b|A,B) = \int_\Lambda P(a|A,\lambda)P(b|B,\lambda)\rho(\lambda)d\lambda$

(all correlations are local correlations)

$\implies$ CHSH inequality is obeyed.

$$|\langle AB\rangle_{\text{lhv}} + \langle AB'\rangle_{\text{lhv}} + \langle A'B\rangle_{\text{lhv}} - \langle A'B'\rangle_{\text{lhv}}| \leq 2$$

On the CHSH inequality in Quantum Mechanics

Consider the **CHSH polynomials**: $\mathcal{B}$ and $\mathcal{B}'$, where

$$\mathcal{B} = AB + AB' + A'B - A'B' \tag{1a}$$

$$\mathcal{B}' = A'B' + A'B + AB' - AB \tag{1b}$$

- Then, all quantum states must obey (Uffink, 2002):

$$\max_{A,A',B,B'} \langle \mathcal{B} \rangle_\rho^2 + \langle \mathcal{B}' \rangle_\rho^2 \leq 8, \quad \forall \rho \in \mathcal{Q}, \tag{2}$$

This implies (and strengthens) the Tsirelson inequality:

$$\max_{A,A',B,B'} |\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq 2\sqrt{2}, \quad \forall \rho \in \mathcal{Q}. \tag{3}$$

Separable states must obey the well-known more stringent bound:

$$\max_{A,A',B,B'} |\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq 2, \quad \forall \rho \in \mathcal{Q}_{\mathrm{sep}}. \tag{4}$$

Orthogonal Measurements

But the maximum bound $2\sqrt{2}$ is **only** obtainable using entangled states and choosing locally *orthogonal measurements*. For qubits the latter are anti-commuting; $\{A, A'\} = 0$, $\{B, B'\} = 0$.

- Now, for any spin-$\frac{1}{2}$ state ρ on $\mathcal{H} = \mathbb{C}^2$, and any orthogonal triple of spin components A, A' and A'' ($A \perp A' \perp A''$), one has

$$\langle A\rangle_\rho^2 + \langle A'\rangle_\rho^2 + \langle A''\rangle_\rho^2 \leq 1. \tag{5}$$

$\Longrightarrow$ But then separable states must obey a sharper quadratic inequality:

$$\max_{A\perp A', B\perp B'} \langle \mathcal{B}\rangle_\rho^2 + \langle \mathcal{B}'\rangle_\rho^2 \leq 2, \quad \forall \rho \in \mathcal{Q}_{\text{sep}}, \tag{6}$$

which in turn gives the linear inequalities:

$$\max_{A\perp A', B\perp B'} |\langle \mathcal{B}\rangle_\rho|, |\langle \mathcal{B}'\rangle_\rho| \leq \sqrt{2}, \quad \forall \rho \in \mathcal{Q}_{\text{sep}}. \tag{7}$$

- ► A factor $\sqrt{2}$ stronger than the original CHSH bound of 2.

On the CHSH inequality

$$\mathcal{B} = AB + AB' + A'B - A'B' \quad , \quad \mathcal{B}' = A'B' + A'B + AB' - AB.$$

$$\langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 8, \qquad \rho \in \mathcal{Q}$$

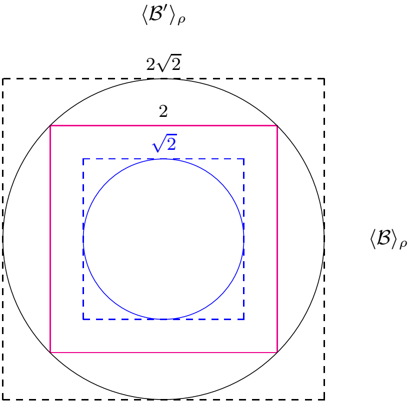
$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2\sqrt{2}, \quad \rho \in \mathcal{Q}$$

$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2, \qquad \rho \in \mathcal{Q}_{\text{sep}}$$

For $A \perp A', B \perp B'$:

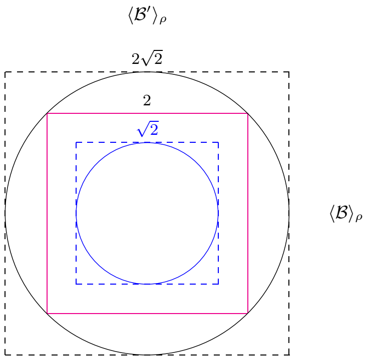
$$\langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 2, \qquad \rho \in \mathcal{Q}_{\text{sep}}$$

$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq \sqrt{2}, \quad \rho \in \mathcal{Q}_{\text{sep}}$$



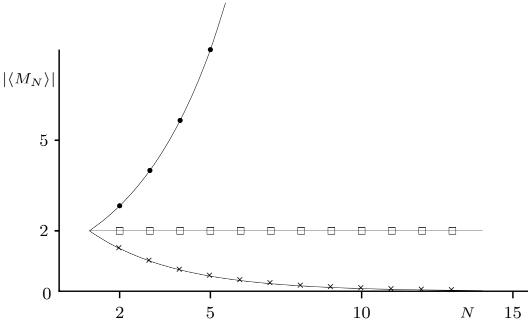
It appears that testing for entanglement within quantum theory and testing quantum mechanics against the class of all LHV theories are not equivalent issues.

- [cf. hidden nonlocality, Werner states] .



All quantum states outside the blue circle, but inside the red square, are entangled. Yet, their correlations are reproducible via a LHV model.

Multipartite generalisation



Here M_N is the Mermin polynomial (a multipartite generalisation of the CHSH polynomial $\mathcal{B}$) for orthogonal observables.

- Usually Bell inequalities are interesting only if there exist a certain quantum state that violates it. However, we here see that it is also very interesting to ask not what quantum states violate a certain Bell inequality, but what quantum states **cannot** violate such a Bell inequality, and by what factor.

- The maximum value of multipartite Bell inequalities obtainable by separable quantum states **exponentially decreases** with respect to the maximum value obtainable by LHV models. Thus as the number of particles increases a larger and larger set of LHV correlations need entanglement to be reproducible by quantum mechanics.

- It is precisely the quantum feature of **incompatible** (i.e. complementary) observables encoded via anticommutativity, which by itself is non-classical, that allows for the strange fact that a '*less than classical*' feature arises in QM.

Further definitions: going beyond QM and CHSH

Surface probabilities: $P(a,b|A,B)$

Determined via measurement of relative frequencies.

Subsurface probabilities: $P(a,b|A,B,\lambda)$

Generally inaccessible, conditioned on hidden variables.

- Definitions of different kinds of bi-partite surface correlations:

a) **Local**: $P(a,b|A,B) = \int_\Lambda d\lambda\, \rho(\lambda)\, P(a|A,\lambda)P(b|B,\lambda)$.

b) **Quantum**: $P(a,b|A,B) =$ Tr$[\, M_a^A \otimes M_b^B\, \rho\,]$, $\sum_a M_a^A = \mathbb{1}$.

c) **No-signaling**: $P(a|A)^B = P(a|A)^{B'} := P(a|A)$

where $P(a|A)^B = \sum_b P(a,b|A,B)$, etc.

d) **Deterministic**: $P(a,b|A,B) \in [0,1]$.

Relating Surface and Subsurface Levels

No theory can be deterministic, non-local and no-signaling.

(i) Any non-local correlation that is no-signaling must be indeterministic, i.e., the outcomes are only probabilistically predicted. (e.g., quantum mechanics, Bohm)

(ii) Any deterministic correlation (i.e., $P(a, b|A, B, \lambda) \in [0, 1]$) that is non-local must be signaling.

Proof: Any deterministic no-signaling correlation must be local. [cf. Masanes et al. (2006)]

(1) Consider a deterministic probability distribution $P_{\rm det}(a,b|AB)$.

$\Longrightarrow$ The outcomes a and b are deterministic functions of A and B: $a = a[A,B]$ and $b = b[A,B]$.

(2) Suppose it is a no-signaling distribution, then

$$P_{\rm det}(a,b|AB) \overset{\rm det}{=} \delta_{(a,b),(a[(A,B],b[A,B])} = \delta_{a,a[A,B]}\delta_{b,b[A,B]}$$
$$= P(a|A,B)P(b|A,B) \overset{\rm ns}{=} P(a|A)P(b|B).$$

This is a local distribution and therefore any deterministic no-signaling correlation must be local.

Determinism, yet indeterminism

Now again consider Bohmian mechanics: because it obeys no-signaling and gives rise to non-local correlations (since it violates the CHSH inequality) it **must** predict the outcomes only probabilistically.

In other words, although fundamentally (at the deeper HV level) deterministic it must necessarily be predictively indeterministic.

► Thus no 'Bohmian demon' can have perfect control over the hidden variables and still be non-local and no-signaling at the surface (as QM requires).

• This is not specific to Bohmian mechanics: **any** deterministic theory that obeys no-signaling and gives non-local correlations must have the same feature: It must predict the outcomes of measurement indeterministically. And this is *independent* of whether the theory is required to reproduce QM.

Discerning no-signaling correlations

We have seen that requiring no-signaling in conjunction with some other constraint has strong consequences.

- But what if we solely require no-signaling? Can we find a non-trivial constraint that follows from no-signaling alone?

The CHSH inequality does not suffice to discern no-signaling correlations because these can maximally violate it up to the algebraic maximum of a value of 4 (e.g., PR-boxes).

► *But an analogue does suffice*:

$$|\langle AB\rangle_{\rm ns} + \langle A'B\rangle_{\rm ns} + \langle A\rangle_{\rm ns} - \langle A'\rangle_{\rm ns}| \leq 2.$$

$\Longrightarrow$ Any correlation that violates this inequality is signaling:
$P(b|B)^A := \sum_a P(a,b|A,B) \neq \sum_a P(a,b|A',B) := P(b|B)^{A'}$.

Reproducing perfect (anti-) correlations

Suppose we want to reproduce the following perfect correlation and perfect anti-correlation using a no-signaling theory:

$$\begin{aligned} &\forall\, \vec{a},\, \vec{b}: \quad \langle \vec{a}\, \vec{b} \rangle = -1, \quad \text{when} \ \ \vec{a} = \vec{b} \\ &\forall\, \vec{a},\, \vec{b}: \quad \langle \vec{a}\, \vec{b} \rangle = 1, \qquad \text{when} \ \ \vec{a} = -\vec{b} \end{aligned}$$

(The singlet state gives such correlations, but we will not assume any quantum mechanics in what follows)

- The no-signaling inequalities give two non-trivial constraints:

$$\langle \vec{a} \rangle^{I}_{\text{ns}} + \langle \vec{a} \rangle^{II}_{\text{ns}} = 0$$

$$\langle -\vec{a} \rangle^{I}_{\text{ns}} = -\langle \vec{a} \rangle^{I}_{\text{ns}}$$

This states that the marginal expectation values for party I and II must add up to zero for measurements in the same direction, and individually they must be odd functions of the settings.

In case both systems are treated the same, i.e., $\langle\vec{a}\rangle^I_{\rm ns} = \langle\vec{a}\rangle^{II}_{\rm ns}$, the marginal expectation values must vanish: $\langle\vec{a}\rangle^I_{\rm ns} = \langle\vec{a}\rangle^{II}_{\rm ns} = 0$.

Thus all marginal probabilities must be uniformly distributed:
$P^I(+|\vec{a}) = P^I(-|\vec{a}) = 1/2$, etc.

Consequently, any no-signaling theory that reproduces perfect (anti-) correlations for all measurement directions and that treats the two systems identically, must locally have uniformly distributed marginals.

► No appeal to QM is needed. If one does, even stronger conclusions can be derived. (Colbeck & Renner; Branciard *et al*.)

Conclusion and Discussion

- The simplest case is not so simple. It still gives rise to a lot of new results, when studied from a larger ‘outside’ point of view.

1. Non-commutativity gives ‘a less than classical effect’: QM generally needs entanglement to reproduce LHV correlations.

2. The conjunction of non-locality and no-signaling (as is the case in QM) is very stringent: the surface probabilities cannot be deterministic. Any determinism must stay beneath the surface.

3. Discerning no-signaling correlations from more general ones can be done via a very similar inequality as the CHSH inequality.

Then, perfect (anti-)correlations for all measurement directions forces a no-signaling theory that treats both particles identically to already have flat local marginals.

Comparing to LHV theories

It is interesting to ask whether one can obtain a similar stronger inequality as $|\langle \mathcal{B} \rangle_\rho| \leq \sqrt{2}$ in the context of local hidden-variable theories, for which we know $|\langle \mathcal{B} \rangle_{\rm lhv}| \leq 2$ (CHSH, 1969) .

The assumption to be added to such an LHV theory is the requirement that for any orthogonal choice of A, A' and A'' and for every given λ we have the analog of (5) which is

$$\langle A \rangle_\lambda^2 + \langle A' \rangle_\lambda^2 + \langle A'' \rangle_\lambda^2 = 1, \qquad (8)$$

where $\langle A \rangle_\lambda = \sum_{a=\pm 1} a\, P(a|A,\lambda)$, etc.

- But a requirement like (8) is *by no means* obvious for a local hidden-variable theory.

Indeed, as has often been pointed out, such a theory may employ a mathematical framework which is completely different from quantum theory.

There is no *a priori* reason why the orthogonality of spin directions should have any particular significance in the hidden-variable theory, and why such a theory should confirm to quantum mechanics in reproducing $\langle A\rangle^2 + \langle A'\rangle^2 + \langle A''\rangle^2 = 1$ if one conditionalizes on a given hidden-variable state λ.

► One is reminded here of Bell's critique on von Neumann's 'no-go theorem'.

Thus, the additional requirement would appear entirely unmotivated within an LHV theory.

(The proof is so short, that for once I believe a proof can enter a presentation.)

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