Monogamy, entanglement and deep hidden variables

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Prospects & Introduction

- I will reconsider the well-known (local) hidden variable program and the famous CHSH inequality.

Some **elementary** investigations and results (by me and others) are presented that have general repercussions.

These are intended to deepen our understanding of what it takes to violate the Bell inequality and how this relates to quantum and no-signalling correlations.

- As part of the recent ‘paradigm change’ to study quantum mechanics (QM) ‘from the outside, not just from the inside’.

Study QM ‘from the outside, not just from the inside’

Motto: In order to understand quantum mechanics it is useful to demarcate those phenomena that are essentially quantum, from those that are more generically non-classical.

Investigate theories that are neither classical nor quantum; explore the space of possible theories from a larger theoretical point of view.

> *“Is quantum mechanics an island in theory space?”* (Aaronson, 2004). If indeed so, where is it?

- It is found that many non-classical properties of QM are generic within the larger family of physical theories.

Thus rather than regard quantum theory special for having these generic ‘quantum’ properties, a better attitude is to regard classical theories as special for not having them.

Methodological morale for this talk:

> *Now it is precisely in cleaning up intuitive ideas for mathematics that one is likely to throw out the baby with the bathwater.*
>
> *J.S. Bell; 'La nouvelle cuisine', 1990.*

► So I will keep things simple, and use a minimal of mathematics.

But the well-known simplest case, the setup of the EPR-Bohm experiments – already studied for over 40 years –, is not so simple after all: there is still very much to be discovered.

Outline

[I] Preliminaries a) Correlations

Surface correlations: $P(A,B|a,b)$

Determined via measurement of relative frequencies.

Subsurface correlations: $P(A,B|a,b,\lambda)$

Generally inaccessible, conditioned on hidden variables λ.

- Definitions of different kinds of bi-partite surface correlations:

a) **Local**: $P(A,B|a,b) = \int_\Lambda d\lambda\, \rho(\lambda)\, P(A|a,\lambda)P(B|b,\lambda)$.

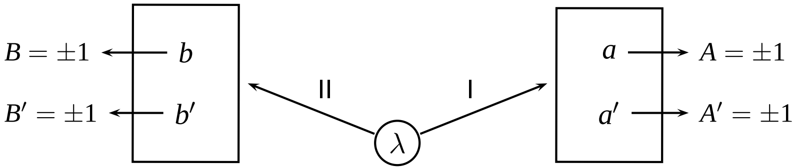
b) **Quantum**: $P(A,B|a,b) =$Tr$[\,M_A^a \otimes M_B^b\, \rho\,]$, $\sum_A M_A^a = \mathbb{1}$.

c) **No-signalling**: $P(A|a)^b = P(A|a)^{b'} := P(A|a)$

where $P(A|a)^b = \sum_B P(A,B|a,b)$, etc.

d) **Deterministic**: $P(A,B|a,b) \in \{0,1\}$.

b) Non-local correlations and Bell's inequality



– 'local causality': $P(A, B|a, b, \lambda) = P(A|a, \lambda)P(B|b, \lambda)$.

– Independence of the Source (IS): $\rho(\lambda|a, b) = \rho(\lambda)$.

- ▶ local causality $\wedge$ IS $\implies P(A, B|a, b) = \int_\Lambda P(A|a, \lambda)P(B|b, \lambda)\rho(\lambda)d\lambda$

  (the correlations are *local*)

- ▶ Consider the Bell-polynomial $\mathcal{B} = ab + ab' + a'b - a'b'$, then

$$|\langle \mathcal{B} \rangle_{\rm lhv}| = |\langle ab \rangle_{\rm lhv} + \langle ab' \rangle_{\rm lhv} + \langle a'b \rangle_{\rm lhv} - \langle a'b' \rangle_{\rm lhv}| \leq 2$$

[II] The CHSH inequality revisited

Consider the **CHSH polynomials**: $\mathcal{B}$ and $\mathcal{B}'$, where

$$\mathcal{B} = ab + ab' + a'b - a'b'$$
$$\mathcal{B}' = a'b' + a'b + ab' - ab$$

- Then, all quantum states must obey (Uffink [2002]):

$$\max_{a,a',b,b'} \langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 8, \quad \forall\rho \in \mathcal{Q},$$

This implies (and strengthens) the Tsirelson inequality:

$$\max_{a,a',b,b'} |\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2\sqrt{2}, \quad \forall\rho \in \mathcal{Q}.$$

Separable states must obey the well-known more stringent bound:

$$\max_{a,a',b,b'} |\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2, \quad \forall\rho \in \mathcal{Q}_{\text{sep}}.$$

Orthogonal Measurements

But the maximum bound $2\sqrt{2}$ is **only** obtainable using entangled states and choosing locally *orthogonal measurements*. For qubits the latter are anti-commuting; $\{a, a'\} = 0$, $\{b, b'\} = 0$.

- Now, for any spin-$\frac{1}{2}$ state ρ on $\mathcal{H} = \mathbb{C}^2$, and any orthogonal triple of spin components a, a' and a'' ($a \perp a' \perp a''$), one has

$$\langle a \rangle_\rho^2 + \langle a' \rangle_\rho^2 + \langle a'' \rangle_\rho^2 \leq 1.$$

$\Longrightarrow$ But then separable states must obey a sharper quadratic inequality:

$$\max_{a \perp a', b \perp b'} \langle \mathcal{B} \rangle_\rho^2 + \langle \mathcal{B}' \rangle_\rho^2 \leq 2, \quad \forall \rho \in \mathcal{Q}_{\text{sep}},$$

which in turn gives the linear inequalities:

$$\max_{a \perp a', b \perp b'} |\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq \sqrt{2}, \quad \forall \rho \in \mathcal{Q}_{\text{sep}}.$$

- A factor $\sqrt{2}$ stronger than the original CHSH bound of 2.

The structure of the CHSH inequality

$\mathcal{B} = ab + ab' + a'b - a'b'$, $\mathcal{B}' = a'b' + a'b + ab' - ab$.

$\langle \mathcal{B} \rangle_\rho^2 + \langle \mathcal{B}' \rangle_\rho^2 \leq 8, \quad \rho \in \mathcal{Q}$

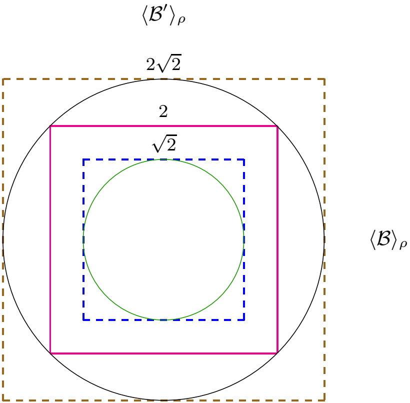
$|\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq 2\sqrt{2}, \quad \rho \in \mathcal{Q}$

$|\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq 2, \quad \rho \in \mathcal{Q}_{\text{sep}}$

For $a \perp a', b \perp b'$:

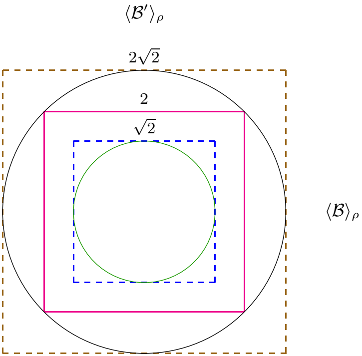
$\langle \mathcal{B} \rangle_\rho^2 + \langle \mathcal{B}' \rangle_\rho^2 \leq 2, \quad \rho \in \mathcal{Q}_{\text{sep}}$

$|\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq \sqrt{2}, \quad \rho \in \mathcal{Q}_{\text{sep}}$



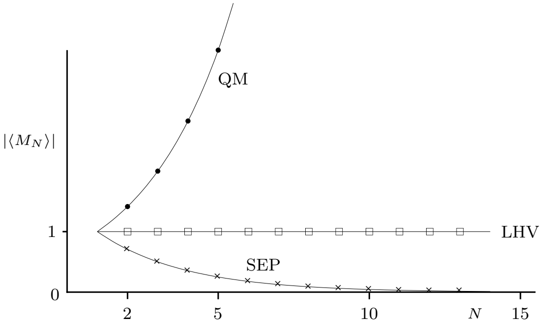
It appears that testing for entanglement within quantum theory and testing quantum mechanics against the class of all LHV theories are not equivalent issues.

- [cf. hidden nonlocality, Werner states] .



For $a \perp a', b \perp b'$: All quantum states outside the green circle, but inside the red square, are entangled. Yet, their correlations are reproducible via a LHV model.

Multipartite generalisation



Here M_N is the Mermin polynomial (a multipartite generalisation of the CHSH polynomial $\mathcal{B}$) for orthogonal observables.

- The maximum value of multipartite Bell inequalities obtainable by separable quantum states **exponentially decreases** with respect to the maximum value obtainable by LHV models.

- Thus as the number of particles increases a larger and larger set of LHV correlations need entanglement to be reproducible by quantum mechanics.

► It is precisely the quantum feature of **incompatible** (i.e. complementary) observables encoded via anticommutativity, which by itself is non-classical, that allows for the strange fact that a '*less than classical*' feature arises in QM.

[III] The power of requiring no-signalling

We have studied local and quantum correlations. What about no-signalling ones?

- No theory can be deterministic, non-local and no-signalling. [cf. Masanes et al. (2006)]

$$\text{deterministic} \wedge \neg\, \text{local} \wedge \text{no-signalling} \implies \lightning$$

(i) Any deterministic non-local correlation must be signalling.

(ii) Any non-local correlation that is no-signalling must be indeterministic, i.e., the outcomes are only probabilistically predicted. (e.g., quantum mechanics, Bohmian mechanics)

Proof: Any deterministic no-signalling correlation must be local. [cf. Masanes et al. (2006)]

(1) Consider a deterministic probability distribution $P_{\mathrm{det}}(A,B|a,b)$.

$\Longrightarrow$ The outcomes A and B are deterministic functions of a and b:

$A = A[a,b]$ and $B = b[A,B]$.

(2) Suppose it is a no-signalling distribution, then

$$P_{\mathrm{det}}(A,B|a,b) \stackrel{\mathrm{det}}{=} \delta_{(A,B),(A[a,b],B[a,b])} = \delta_{A,A[a,b]}\delta_{B,B[a,b]}$$

$$= P(A|a,b)P(B|a,b) \stackrel{\mathrm{ns}}{=} P(A|a)P(B|b).$$

This is a local distribution and therefore any deterministic no-signalling correlation must be local.

Determinism, yet indeterminism

Now again consider Bohmian mechanics: because it obeys no-signalling and gives rise to non-local correlations (since it violates the CHSH inequality) it **must** predict the outcomes only probabilistically.

In other words, although fundamentally (at the deeper HV level) deterministic it must necessarily be predictively indeterministic.

- Thus no 'Bohmian demon' can have perfect control over the hidden variables and still be non-local and no-signalling at the surface (as QM requires).

- This is not specific to Bohmian mechanics: **any** deterministic theory that obeys no-signalling and gives non-local correlations must have the same feature. And this is *independent* of whether the theory is required to reproduce QM.

Discerning no-signalling correlations

We have seen that requiring no-signalling in conjunction with some other constraint has strong consequences.

- But what if we solely require no-signalling? Can we find a non-trivial constraint that follows from no-signalling alone?

The CHSH inequality does not suffice to discern no-signalling correlations because these can give violations up to the algebraic maximum of a value of 4 (e.g., PR-boxes).

- *But an analogue does suffice*:

$$|\langle ab\rangle_{\text{ns}} + \langle a'b\rangle_{\text{ns}} + \langle a\rangle_{\text{ns}} - \langle b'\rangle_{\text{ns}}| \leq 2.$$

$\Longrightarrow$ Any correlation that violates this inequality is signalling:

$P(B|b)^a := \sum_A P(A,B|a,b) \neq \sum_A P(A,B|a',b) := P(B|b)^{a'}$.

[IV] Shareability and monogamy of states and correlations

What are the structural limitations in the way parts and wholes can be configurated according to physical theories?

- Here one focuses on the limitations set by physical theories on the shareability of subsystem states, and of the correlations present in a composite system.

Alternatively, can we create particular composite systems (in particular configurations) by sharing/duplicating a subsystem, while maintaining the original configuration (of physical states and/or correlations) between the initial subsystems?

- If this is not possible this is referred to as 'monogamy'.

Note: 'sharing' is to be understood kinematical, not dynamical.

(i) Monogamy of quantum states

Entanglement is monogamous

If a pure quantum state of two systems is entangled, then none of the two systems can be entangled with a third system.

1. Suppose that systems $\mathfrak{a}$ and $\mathfrak{b}$ are in a pure entangled state.
2. Then when the system $\mathfrak{ab}$ is considered as part of a larger system, the reduced density operator for $\mathfrak{ab}$ must by assumption be a pure state.
3. However, for the composite system $\mathfrak{ab}$ (or for any of its subsystems $\mathfrak{a}$ or $\mathfrak{b}$) to be entangled with another system, the reduced density operator of $\mathfrak{ab}$ must be a mixed state.
4. But since it is by assumption pure, no entanglement between $\mathfrak{ab}$ and any other system can exist.

Mixed state entanglement can be shared

The W-state $|\psi\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ has bi-partite reduced states that are all identical and entangled.

► 'sharing of mixed state entanglement', or 'promiscuity of entanglement'.

But this promiscuity is not unbounded: no entangled bi-partite state can be shared with an infinite number of parties.

Here a bi-partite state $\rho_{\mathfrak{ab}}$ is said to be <u>N-shareable</u> when it is possible to find a quantum state $\rho_{\mathfrak{ab}_1\mathfrak{b}_2\ldots\mathfrak{b}_N}$ such that

$$\rho_{\mathfrak{ab}} = \rho_{\mathfrak{ab}_1} = \rho_{\mathfrak{ab}_2} = \ldots = \rho_{\mathfrak{ab}_N},$$

where $\rho_{\mathfrak{ab}_k}$ is the reduced state for parties $\mathfrak{a}$ and $\mathfrak{b}_k$.

- Fannes *et al*. [1988], Raggio *et al*. [1989]: A bi-partite state is N-shareable for all N (also called ∞-shareable) iff it is separable.

Quantifying the monogamy of entanglement

Coffman, Kundu and Wootters [2000] gave a trade-off relation between how entangled $\mathfrak{a}$ is with $\mathfrak{b}$, and how entangled $\mathfrak{a}$ is with $\mathfrak{c}$ in a three-qubit system $\mathfrak{abc}$ that is in a pure state $|\psi\rangle$:

$$\tau(\rho_{\mathfrak{ab}}) + \tau(\rho_{\mathfrak{ac}}) \leq 4\det\rho_{\mathfrak{a}}$$

with $\rho_{\mathfrak{a}} = \mathrm{Tr}_{\mathfrak{bc}}[|\psi\rangle\langle\psi|]$ and where $\tau(\rho_{\mathfrak{ab}})$ is the tangle between $\mathfrak{a}$ and $\mathfrak{b}$ (analogous for $\tau(\rho_{\mathfrak{ac}})$).

The multi-partite generalization has been recently proven by Osborne & Verstraete [2006].

(ii) Monogamy of non-local correlations

Suppose one has some no-signalling three-party probability distribution $P(A_1, A_2, A_3 | a_1, a_2, a_3)$ for parties $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$.

- If the marginal distribution $P(A_1, A_2 | a_1, a_2)$ for $\mathfrak{ab}$ is extremal (a vertex of the no-signalling polytope), then:

$$P(A_1, A_2, A_3 | a_1, a_2, a_3) = P(A_1, A_2 | a_1, a_2)\, P(A_3 | a_3).$$

This implies that party $\mathfrak{c}$ is completely uncorrelated with party $\mathfrak{ab}$!

Thus the extremal correlation $P(A_1, A_2 | a_1, a_2)$ is completely *monogamous*.

Quantifying the monogamy of non-local correlations

Extremal no-signalling correlations thus show monogamy, but what about non-extremal no-signalling correlations?

► Just as was the case for quantum states where non-extremal (mixed state) entanglement can be shared, non-extremal no-signalling correlations can be shared as well.

- Toner [2006] proved a tight trade-off relation:

$$|\langle \mathcal{B}_{\mathfrak{ab}} \rangle_{\mathrm{ns}}| + |\langle \mathcal{B}_{\mathfrak{ac}} \rangle_{\mathrm{ns}}| \leq 4.$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are different parties.

Extremal no-signalling correlations can attain $|\langle \mathcal{B}_{\mathfrak{ab}} \rangle_{\mathrm{ns}}| = 4$ so that necessarily $|\langle \mathcal{B}_{\mathfrak{ac}} \rangle_{\mathrm{ns}}| = 0$, and vice versa. But non-extremal ones are shareable.

Monogamy for other kinds of correlations

$$\mathcal{B}_{\mathfrak{ab}} = ab + ab' + a'b - a'b' \quad , \quad \mathcal{B}_{\mathfrak{ac}} = ac + ac' + a'c - a'c'$$

- For general unrestricted correlations no monogamy holds, i.e., $|\langle\mathcal{B}_{\mathfrak{ab}}\rangle|$ and $|\langle\mathcal{B}_{\mathfrak{ac}}\rangle|$ are not mutually constrained.
- Quantum correlations are monogamous: $\langle\mathcal{B}_{\mathfrak{ab}}\rangle^2_{\rm qm} + \langle\mathcal{B}_{\mathfrak{ac}}\rangle^2_{\rm qm} \leq 8$.
- Classical correlations are not monogamous. It is possible to have both $|\langle\mathcal{B}_{\mathfrak{ab}}\rangle_{\rm lhv}| = 2$ and $|\langle\mathcal{B}_{\mathfrak{ac}}\rangle_{\rm lhv}| = 2$.
- Separable quantum state are neither monogamous: $|\langle\mathcal{B}_{\mathfrak{ab}}\rangle_{\rm qm}|, |\langle\mathcal{B}_{\mathfrak{ac}}\rangle_{\rm qm}| \leq 2$, $\rho \in \mathcal{Q}_{\rm sep}$.

(For orthogonal measurements a stronger bound holds: $\leq \sqrt{2}$)

Monogamy of correlations

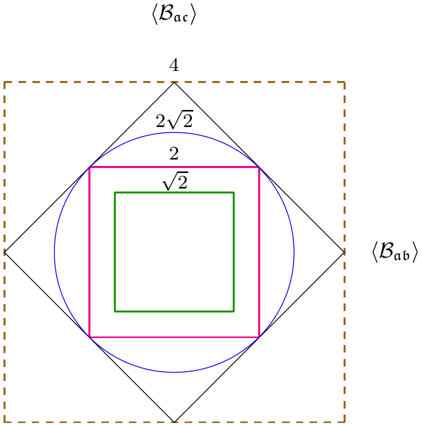
$$\mathcal{B}_{\mathfrak{ab}} = ab + ab' + a'b - a'b' \quad , \quad \mathcal{B}_{ac} = ac + ac' + a'c - a'c'$$

$|\langle\mathcal{B}_{\mathfrak{ab}}\rangle|,\ |\langle\mathcal{B}_{\mathfrak{ac}}\rangle| \leq 4$

$|\langle\mathcal{B}_{\mathfrak{ab}}\rangle_{\rm ns}| + |\langle\mathcal{B}_{\mathfrak{ac}}\rangle_{\rm ns}| \leq 4$ [a]

$\langle\mathcal{B}_{\mathfrak{ab}}\rangle^2_{\rm qm} + \langle\mathcal{B}_{\mathfrak{ac}}\rangle^2_{\rm qm} \leq 8 \quad \rho \in \mathcal{Q}$ [b]

$|\langle\mathcal{B}_{\mathfrak{ab}}\rangle_{\rm lhv}|,\ |\langle\mathcal{B}_{\mathfrak{ac}}\rangle_{\rm lhv}| \leq 2$



For $a \perp a', b \perp b', c \perp c'$:

$|\langle\mathcal{B}_{\mathfrak{ab}}\rangle_{\rm qm}|,\ |\langle\mathcal{B}_{\mathfrak{ac}}\rangle_{\rm qm}| \leq \sqrt{2} \quad \rho \in \mathcal{Q}_{\rm sep}$ [c]

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[a] Toner [2006]

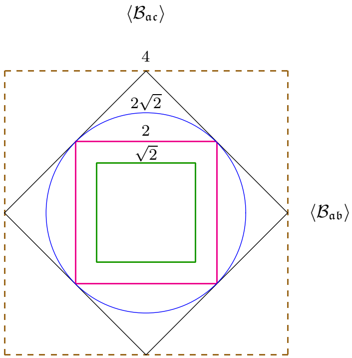
[b] Toner & Verstraete [2006]; stronger state-dependent bound (MPS [2008])

[c] MPS [2007]

Consequences of this monogamy of correlations

In case the no-signalling correlations are non-local they can not be shared (it is impossible that both $|\langle \mathcal{B}_{\mathfrak{ab}} \rangle_{\text{ns}}| \geq 2$ and $|\langle \mathcal{B}_{\mathfrak{ac}} \rangle_{\text{ns}}| \geq 2$).

- The monogamy bound allows for discriminating no-signalling from general correlations: if the bound is violated the correlations must be signalling.
- Extremal quantum and no-signalling correlations are fully monogamous.
- This allows for secure key-distribution protocols that are based on the laws of physics only (and not on some computationally hard procedure).



local realism $\iff$ ∞-shareability of correlations

$\exists$ local model for $P(A, B|a, b)$ when party 1 has an arbitrary number and party 2 has N possible measurements

$\iff$

N-shareability of correlations

- Proof (Masanes, *et al* [2006]):

$\implies$ classical information can be cloned indefinitely.

$\impliedby$ Since $P(A, B|a, b)$ is shareable to N parties (labelled $\mathfrak{b}_i$, $i = 1, \ldots, N$), the correlations between a performed on party 1 and b_i on party 2 are the same as the correlations between measurements of a on party 1 and b_i on the extra party $\mathfrak{b}_i$.

Therefore, the N measurements $b_1, \ldots, b_N$ performed by party 2 can be viewed as one large measurement performed on the N parties $\mathfrak{b}_i$ ($i = 1, \ldots, N$). Lastly, there always exists a local hidden variable model when one of the two parties has only one measurement.

[V] Interpreting Bell's Theorem

► Schumacher [2008]: Bell's theorem is about the shareability of correlations; its real physical message is *not* about local realism, since we don't need ∞-shareable (i.e., local realism) to obtain the CHSH inequality that quantum mechanics violates.

• Claim: 2-shareability is sufficient to obtain the CHSH inequality ; and this is a weaker claim than the assumption of local realism.

2-shareability implies CHSH inequality

Consider an EPR-Bohm setup for parties 1 and 2.

Assume that all possible correlations between 1 and 2 are shareable to another party $1'$ and $2'$ that conceivably exist. Then for the outcomes:

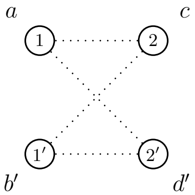
$$A(C+D')+B'(C-D')=\pm 2$$

which implies for the expectation values

$$|\langle ac\rangle+\langle ad'\rangle+\langle b'c\rangle-\langle b'd'\rangle|\leq 2$$

2-shareability implies

$$|\langle ac\rangle+\langle ad\rangle+\langle bc\rangle-\langle bd\rangle|\leq 2$$



The shareability is supposed to justify the counterfactual reasoning.

Critique

1) Despite Schumacher's argument, it is indeed still the case that quantum mechanics is non-local in the sense that some quantum correlations cannot be given a factorisable form in terms of local correlations.

2) The argument is not logically weaker than standard derivations of Bell's theorem. We only need to assume that local realism holds *just* for measurement of four different observables: two for party 1 (e.g., a, a') and two for party 2 (e.g., b, b').

It is thus not necessary to assume full blown local realism for an *unlimited* number of observables and parties.

[VI] Discussion

- The simplest case is not so simple. It still gives rise to a lot of new results, when studied from a larger 'outside' point of view.

1. Non-commutativity gives 'a less than classical effect': QM generally needs entanglement to reproduce LHV correlations.

2. The conjunction of non-locality and no-signalling (as is the case in QM) is very stringent: the surface probabilities cannot be deterministic. Any determinism must stay beneath the surface.

3. Discerning no-signalling correlations from more general ones can be done via a very similar inequality as the CHSH inequality.

4. Non-local correlations (whether quantum or no-signalling) are monogamous, whereas ∞-shareability $\iff$ local realism. (Quantumcryptography ; a new view on Bell's theorem?)