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Partial separability and entanglement criteria for multiqubit quantum states

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We explore the subtle relationships between partial separability and entanglement of subsystems in multiqubit quantum states and give experimentally accessible conditions that distinguish between various classes and levels of partial separability in a hierarchical order. These conditions take the form of bounds on the correlations of locally orthogonal observables. Violations of such inequalities give strong sufficient criteria for various forms of partial inseparability and multiqubit entanglement. The strength of these criteria is illustrated by showing that they are stronger than several other well-known entanglement criteria (the fidelity criterion, violation of Mermin-type separability inequalities, the Laskowski-Żukowski criterion, and the Dür-Cirac criterion) and also by showing their great noise robustness for a variety of multiqubit states, including N-qubit Greenberger-Horne-Zeilinger states and Dicke states. Furthermore, for $N\geqslant 3$ they can detect bound entangled states. For all these states, the required number of measurement settings for implementation of the entanglement criteria is shown to be only $N+1$. If one chooses the familiar Pauli matrices as single-qubit observables, the inequalities take the form of bounds on the antidiagonal matrix elements of a state in terms of its diagonal matrix elements.

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I. INTRODUCTION

The problem of characterizing entanglement for multipartite quantum systems has recently drawn much attention. An important issue in this problem is that, apart from the extreme cases of full separability and full entanglement of all particles in the system, one also has to face the intermediate cases in which only some particles in the system are entangled and others not. The latter states are usually called "partially separable" or, more precisely, "k-separable" when they take the form of a mixture of states that factorize when the N-partite system is partitioned into k subsystems ($k \leqslant N$) [1–4]. In this paper we will focus on multiqubit systems only. We propose a classification of partially separable states for such systems, slightly extending the classification introduced by Dür and Cirac [2]. This classification consists of a hierarchy of levels corresponding to the k-separable states for $k=1,\ldots,N$, and within each level various classes are distinguished by specifying under which partitions of the system the state is separable or not.

Several experimentally accessible conditions to characterize k-separable multiqubit states have already been proposed, e.g., by Laskowski and Żukowski [5], Mermin-type separability inequalities [1,6–10], or in terms of entanglement witnesses [11]. However, these conditions do not distinguish the various classes within the levels. Separability conditions that do distinguish some of these classes in the hierarchy were developed by Dür and Cirac. Here we present separability conditions that extend and strengthen all the conditions just mentioned.

These conditions take the form of sets of inequalities that bound the correlations for standard Bell-type experiments (involving at each site measurement of two orthogonal spin observables). They form a hierarchy with bounds that decrease by a factor of 4 for each level k in the partial separability hierarchy. For the classes within a given level, the inequalities give state-dependent bounds, differing for each class. Violations of the inequalities provide strong sufficient criteria for various forms of inseparability and multiqubit entanglement.

We demonstrate the strength of these conditions in two ways: First, by showing that they imply several other general separability conditions, namely the fidelity criterion [12–14], the partial separability conditions just mentioned, i.e., the Laskowski-Żukowski condition (with a strict improvement for $k=2$, N), the Dür-Cirac condition, and the Mermin-type separability inequalities. We also show that the latter are equivalent to the Laskowski-Żukowski condition.

Second, we compare the conditions to other state-specific multiqubit entanglement criteria [11,15,16] both for their white noise robustness and for the number of measurement settings required in their implementation. In particular, we show (i) detection of bound entanglement for $N\geqslant 3$ with noise robustness for detecting the bound entangled states of Ref. [3] that goes to 1 for large N (i.e., maximal noise robustness), (ii) detection of the four qubit Dicke state with noise robustness 0.84 and 0.36 for detecting it as entangled and fully entangled, respectively, (iii) great noise and decoherence robustness [17,18] in detecting entanglement of the N-qubit Greenberger-Horne-Zeilinger (GHZ) state where for colored noise and for decoherence due to dephasing the robustness for detecting full entanglement goes to 1 for large N, and lastly, (iv) better white noise robustness than the stabilizer witness criteria of Ref. [11] for detecting the N-qubit GHZ states. In all these cases it is shown that only $N+1$ settings are needed.

Choosing the familiar Pauli matrices as the local orthogonal observables yields a convenient matrix element representation of the partial separability conditions. In this representation, the inequalities give specific bounds on the antidiagonal matrix elements in terms of the diagonal ones.

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Further, some comments will be made along the way on how these results relate to the original purpose [19] of Bell-type inequalities to test local hidden-variable (LHV) models against quantum mechanics. Most notably, when the number of parties is increased, there is not only an exponentially increasing factor that separates the correlations allowed in maximally entangled states in comparison to those of local hidden-variable theories, but, surprisingly, also an exponentially increasing factor between the correlations allowed by LHV models and those allowed by nonentangled qubit states.

This paper is structured as follows. In Sec. II we define the relevant partial separability notions and extend the hierarchic partial separability classification of Ref. [2]. There we also introduce the notions of k-separable entanglement and of m-partite entanglement in order to investigate the relation between partial separability and multipartite entanglement. We then discuss four known partial separability conditions discussed above. In Sec. III we derive partial separability conditions for N qubits in terms of locally orthogonal observables. They provide the desired necessary conditions for the full hierarchic separability classification. In Sec. IV the experimental strength of these criteria is discussed. We end in Sec. V with a discussion of the results obtained.

II. PARTIAL SEPARABILITY AND MULTIPARTITE ENTANGLEMENT

In this section we introduce terminology and definitions to be used in later sections. We define the notions of k-separability, α_k-separability, k-separable entanglement, and m-partite entanglement and use these notions to capture aspects of the separability and entanglement structure in multipartite states. We review the separability hierarchy introduced by Dür and Cirac [2] and extend their classification. We also discuss four partial separability conditions known in the literature. These conditions will be strengthened in Sec. III.

A. Partial separability and the separability hierarchy

Consider an N-qubit system with Hilbert space $\mathcal{H}=\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Let $\alpha_k=(S_1,\ldots,S_k)$ denote a partition of $\{1,\ldots,N\}$ into k disjoint nonempty subsets ($k \leqslant N$). Such a partition corresponds to a division of the system into k distinct subsystems, also called a k-partite split [2]. A quantum state ρ of this N-qubit system is k-separable under a specific k-partite split α_k [1–4] if and only if it is fully separable in terms of the k subsystems in this split, i.e., if and only if

$$\rho = \sum_i p_i \otimes_{n=1}^{k} \rho_i^{S_n}, \quad p_i \geqslant 0, \quad \sum_i p_i = 1, \tag{1}$$

where ρ^{S_n} is a state of subsystem corresponding to S_n in the split α_k. We denote such states as $\rho \in \mathcal{D}_N^{\alpha_k}$ and also call them α_k-separable, for short. Clearly, $\mathcal{D}_N^{\alpha_k}$ is a convex set. A state of the N-qubit system outside this set is called α_k-inseparable.

More generally, a state ρ is called k-separable [5,20–23] (denoted as $\rho \in \mathcal{D}_N^{k\text{-sep}}$) if and only if there exists a convex decomposition

$$\rho = \sum_j p_j \otimes_{n=1}^{k} \rho^{S_n^{(j)}}, \quad p_j \geqslant 0, \quad \sum_j p_j = 1, \tag{2}$$

where each state $\otimes_{n=1}^{k} \rho^{S_n^{(j)}}$ is a tensor product of k density matrices of the subsystems corresponding to some such partition $\alpha_k^{(j)}$, i.e., it factorizes under this split $\alpha_k^{(j)}$. In this definition, the partition may vary for each j, as long as it is a k-partite split, i.e., contains k disjoint nonempty sets. Clearly $\mathcal{D}_N^{k\text{-sep}}$ is also convex; it is the convex hull of the union of all $\mathcal{D}_N^{\alpha_k}$ for fixed values of k and N. States that are not k-separable will be called k-inseparable. Note that a k-separable state need not be α_k-separable for any particular split α_k [24]; and even the converse implication need not hold: If a state is biseparable under every bipartition, it does not have to be fully separable, as shown by the three-partite examples in Ref. [25]. Similar observations (using different terminology) were presented in Refs. [20,21], but below we will present a more systematic investigation.

The notion of k-separability naturally induces a hierarchic ordering of the N-qubit states. Indeed, the sequence of sets $\mathcal{D}_N^{k\text{-sep}}$ is nested: $\mathcal{D}_N^{N\text{-sep}} \subset \mathcal{D}_N^{(N-1)\text{-sep}} \subset \cdots \subset \mathcal{D}_N^{1\text{-sep}}$. In other words, k-separability implies ℓ-separability for all $\ell \leqslant k$. We call a k-separable state that is not $(k+1)$-separable "k-separable entangled." Thus each N-qubit state can be characterized by the level k for which it is k-separable entangled, and these levels provide a hierarchical ranking: at one extreme end are the 1-separable entangled states which are fully entangled (e.g., the GHZ states), at the other end are the N-separable or fully separable states (e.g., product states or the "white noise state" $\mathbb{1}/2^N$).

Often, it is interesting to know how many qubits are entangled in a k-separable entangled state. However, this question does not have a unique answer. For example, take $N=4$ and $k=2$ (biseparability). In this case two types of states may occur in the decomposition (2), namely $\rho^{\{ij\}} \otimes \rho^{\{kl\}}$ and $\rho^{\{i\}} \otimes \rho^{\{jkl\}}$ $(i,j,k,l=1,2,3,4)$. A 2-separable entangled four-partite state might thus be two- or three-partite entangled.

In general, an N-qubit state ρ will be called "m-partite entangled" if and only if a decomposition of the state such as in Eq. (2) exists such that each subset $S^{(i)}$ contains at most m parties, but no such decomposition is possible when all the k subsets are required to contain less than m parties [13]. [In Refs. [20,21] this is called "not producible by $(m-1)$-partite entanglement."] It follows that a k-separable entangled state is also m-partite entangled, with $\lceil N/k \rceil \leqslant m \leqslant N-k+1$. Here $\lceil N/k \rceil$ denotes the smallest integer which is not less than N/k. Thus a state that is k-separably entangled $(k<N)$ is at least $\lceil N/k \rceil$-partite entangled and might be up to $(N-k+1)$-partite entangled. Therefore conditions that distinguish k-separability from $(k+1)$-separability also provide conditions for m-partite entanglement, but generally allowing a wide range of values of m. For example, for $N=100$ and $k=2$, m might lie anywhere between 50 and 99.

Of course, a much tighter conclusion about m-partite entanglement can be drawn if we know exactly under which splits the state is separable. This is why the notion of α_k-separability is helpful, since it provides these finer distinctions. For example, suppose that a 100-qubit state is separable under the bipartite split ({1},{2,…,100}) but under

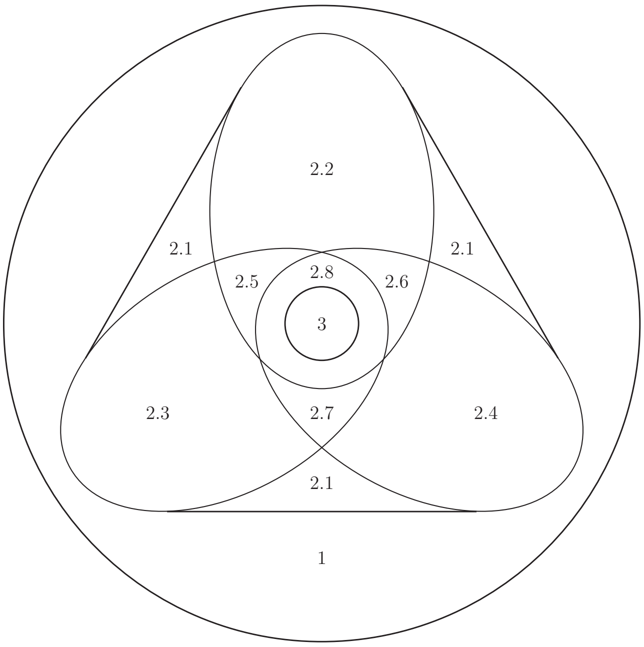

FIG. 1. Schematic representation of the ten partial separability classes of three-qubit states.

no other bipartite split. This state would then be 2-separable (biseparable) but now we could also infer that $m=99$. On the other hand, if the state were only separable under the split $(\{1,\ldots,50\},\{51,\ldots,100\})$, it would still be biseparable, but only m-partite entangled for $m=50$.

Dür and Cirac [2] provided such a fine-grained classification of N-qubit states by considering their separability or inseparability under all k-partite splits. Let us introduce this classification (with a slight extension) by means of the example of three qubits, labeled as a,b,c.

Class 3. Starting with the lowest level $k=3$, there is only one 3-partite split, a-b-c, and consequently only one class to be distinguished at this level, i.e., $\mathcal{D}_3^{a\text{-}b\text{-}c}$. This set coincides with $\mathcal{D}_3^{3\text{-sep}}$.

Classes 2.1–2.8. Next, at level $k=2$, there are three bipartite splits: a-(bc), b-(ac), and c-(ab) which define the sets $\mathcal{D}_3^{a\text{-}(bc)}$, $\mathcal{D}_3^{b\text{-}(ac)}$, and $\mathcal{D}_3^{c\text{-}(ab)}$. One can further distinguish classes defined by all logical combinations of separability and inseparability under these splits, i.e., all the set-theoretical intersections and complements shown in Fig. 1. This leads to classes 2.2–2.8. Dür and Cirac showed that all these classes are nonempty. To these, we add one more class 2.1: the set of biseparable states that are not separable under any split. As we have seen, this set is nonempty too.

Class 1. Finally, at level $k=1$ there is again only one (trivial) split (abc), and thus only one class, consisting of all the fully entangled states, i.e., $\mathcal{D}_3^{1\text{-sep}}\setminus\mathcal{D}_3^{2\text{-sep}}$.

We feel that the above extension is desirable since otherwise the Dür-Cirac classification would not distinguish between class 2.1 and class 1. However, states in class 2.1 are simply convex combinations of states that are biseparable under different bipartite splits. Such states can be realized by mixing the biseparable states and are conceptually different from the fully inseparable states of class 1.

This three-partite example serves to illustrate how the Dür-Cirac separability classification works for general N. Level k $(1\leqslant k\leqslant N)$ of the separability hierarchy consists of all k-separable entangled states. Each level is further divided into distinct classes by considering all logically possible combinations of separability and inseparability under the various k-partite splits. The number of such classes increases rapidly with N, and therefore we will not attempt to list them. In general, all such classes may be nonempty. As an extension of the Dür-Cirac classification, we distinguish at each level $1<k<N$ one further class, consisting of k-separable entangled states that are not separable under any k-partite split.

In order to find relations between these classes, the notion of a *contained split* is useful [2]. A k-partite split α_k is contained in a l-partite split α_l, denoted as $\alpha_k\prec\alpha_\ell$, if α_l can be obtained from α_k by joining some of the subsets of α_k. The relation $\prec$ defines a partial order between splits at different levels. This partial order is helpful because α_k-separability implies α_ℓ-separability of all splits α_ℓ containing α_k. We will use this implication below to obtain conditions for separability of a k-partite split at level k from such conditions on all $(k-1)$-partite splits at level $k-1$ this k-partite split is contained in. Conditions at a lower level thus imply conditions at a higher level.

The multipartite entanglement properties of k-separable or α_k-separable states are subtle, as can be seen from the following examples.

(i) Mixing states does not conserve m-partite entanglement. Take $N=3$, then mixing the 2-partite entangled 2-separable states $|0\rangle\otimes(|00\rangle+|11\rangle)/\sqrt{2}$ and $|0\rangle\otimes(|00\rangle-|11\rangle)/\sqrt{2}$ with equal weights gives a 3-separable state $(|000\rangle\langle000|+|011\rangle\langle011|)/2$.

(ii) An N-partite state can be m-partite entangled $(m<N)$ even if it has no m-partite subsystem whose (reduced) state is m-partite entangled [13,20]. Such states are said to have irreducible m-partite entanglement [26]. Thus a state of which some reduced state is m-partite entangled is itself at least m-partite entangled, but the converse need not be true.

(iii) Consider a biseparable entangled state that is only separable under the bipartite split $(\{1\},\{2,\ldots,N\})$. One cannot infer that the subsystem $\{2,\ldots,N\}$ is $(N-1)$-partite entangled. A counterexample is the three-qubit state $\rho=(|0\rangle\langle0|\otimes P_-^{(bc)}+|1\rangle\langle1|\otimes P_+^{(bc)})/2$ which is biseparable only under the partition a-(bc), and thus bipartite entangled, but has no bipartite subsystem whose reduced state is entangled. Here $P_+^{(bc)}$ and $P_-^{(bc)}$ denote projectors on the Bell states $|\psi^{\pm}\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$ for parties b and c, respectively.

(iv) A state that is inseparable under all splits but which is not fully inseparable (i.e., $\rho\in\mathcal{D}_N^{k\text{-sep}}$ with $k>1$ and $\rho\notin\cup_{\alpha_k}\mathcal{D}_N^{\alpha_k},\ \forall\,\alpha_k,k)$ might still have all forms of m-partite entanglement apart from full entanglement, i.e., it could be m-partite entangled with $2\leqslant m\leqslant N-1$. Thus the state could even have m-partite entanglement as low as 2-partite entanglement, although it is inseparable under all splits. For example, Tóth and Gühne [21] consider a mixture of two N-partite states where each of them is $(\lceil N/2\rceil)$-separable according to different splits. This mixed state is by construction $(\lceil N/2\rceil)$-separable, not biseparable under any split, yet only 2-partite entangled. See also the example in [24] which is

$(N-1)$-separable and only 2-partite entangled.

(v) Lastly, N-partite fully entangled states exist where no m-partite reduced state is entangled (such as N-qubit GHZ state) and also where all m-partite reduced states are entangled (such as the N-qubit W states) [27].

These examples serve to emphasize that one should be very cautious in inferring the existence of entanglement in subsystems of a larger system which is known to be m-partite entangled or k-separable entangled for some specific value of m and k.

B. Separability conditions

We now review four separability conditions for qubits, which will all be strengthened in the next section. These are necessary conditions for states to be k-separable, 2-separable, and α_k-separable, respectively.

(1) Laskowski and Żukowski [5] showed that for any k-separable N-qubit state ρ the antidiagonal matrix elements (denoted by $\rho_{j,\bar{j}}$, where $\bar{j}=d+1-j$, $d=2^N$) must satisfy

$$\max_j|\rho_{j,\bar{j}}| \leq \left(\frac{1}{2}\right)^k, \quad \forall\ \rho \in \mathcal{D}_N^{k\text{-sep}}. \tag{3}$$

This condition can be easily proven by the observation that for any density matrix to be physically meaningful its antidiagonal matrix elements must not exceed $1/2$. Therefore antidiagonal elements of a product of k density matrices cannot be greater than $(1/2)^k$. By convexity, this results then holds all k-separable states. Note that this condition is not basis dependent.

It follows from Eq. (3) that if the antidiagonal matrix elements of state ρ obey

$$\left(\frac{1}{2}\right)^k \geq \max_j|\rho_{j,\bar{j}}| > \left(\frac{1}{2}\right)^{k+1}, \tag{4}$$

then ρ is at most k-separable, i.e., k-separable entangled, and thus at least m-partite entangled, with $m \geq [N/k]$.

The partial separability condition (3) does not yet explicitly refer to directly experimentally accessible quantities. However, in the next section we will rewrite this condition in terms of expectation values of local observables and show that they are equivalent to Mermin-type separability inequalities.

(2) Mermin-type separability inequalities [1,6,8–10]. Consider the familiar Clauser-Horne-Shimony-Holt operator for two qubits (labeled as a and b) which is defined by

$$M^{(2)} := X_a \otimes X_b + X_a \otimes Y_b + Y_a \otimes X_b - Y_a \otimes Y_b. \tag{5}$$

Here, X_a and Y_a denote two spin observables on the Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ of qubit a and b. The so-called Mermin operator [28] is a generalization of this operator to N qubits [labeled as $(a,b,\dots,n)$], defined by the recursive relation

$$M^{(N)} := \frac{1}{2}M^{(N-1)} \otimes (X_n + Y_n) + \frac{1}{2}M'^{(N-1)} \otimes (X_n - Y_n), \tag{6}$$

where M' is the same operator as M but with all X's and Y's interchanged.

In the special case where, for each qubit, the spin observables X and Y are orthogonal, i.e., $\{X_i, Y_i\}=0$ for $i \in \{a,\dots,n\}$, Nagata *et al.* [1] obtained the following k-separability conditions:

$$\langle M^{(N)}\rangle^2 + \langle M'^{(N)}\rangle^2 \leq 2^{(N+3)}\left(\frac{1}{4}\right)^k, \quad \forall\ \rho \in \mathcal{D}_N^{k\text{-sep}}. \tag{7}$$

As just mentioned, the next section will show that these inequalities are equivalent to the Laskowski-Żukowski inequalities. The quadratic inequalities (7) also imply the following sharp linear Mermin-type inequality for k-separability:

$$|\langle M^{(N)}\rangle| \leq 2^{(N+3)/2}\left(\frac{1}{2}\right)^k, \quad \forall\ \rho \in \mathcal{D}_N^{k\text{-sep}}. \tag{8}$$

For $k=N$ inequality (8) reproduces a result obtained by Roy [10].

(3) The fidelity $F(\rho)$ of a N-qubit state ρ with respect to the generalized N-qubit GHZ state $|\Psi^N_{\mathrm{GHZ},\alpha}\rangle := (|0\rangle^{\otimes N} + e^{i\alpha}|1\rangle^{\otimes N})/\sqrt{2}$ $(\alpha \in \mathbb{R})$ is defined as

$$F(\rho) := \max_\alpha \langle\Psi^N_{\mathrm{GHZ},\alpha}|\rho|\Psi^N_{\mathrm{GHZ},\alpha}\rangle = \frac{1}{2}(\rho_{1,1} + \rho_{d,d}) + |\rho_{1,d}|, \tag{9}$$

The fidelity condition [12–14] (also known as the projection-based witness [11]) says that for all biseparable ρ,

$$F(\rho) \leq 1/2, \quad \forall\ \rho \in \mathcal{D}_N^{2\text{-sep}}. \tag{10}$$

In other words, $F(\rho) > 1/2$ is a sufficient condition for full N-partite entanglement. An equivalent formulation of Eq. (10) is

$$2|\rho_{1,d}| \leq \sum_{j\neq 1,d} \rho_{j,j}, \quad \forall\ \rho \in \mathcal{D}_N^{2\text{-sep}}. \tag{11}$$

Of course, analogous conditions may be obtained by replacing $|\Psi^N_{\mathrm{GHZ},\alpha}\rangle$ in the definition (9) by any other maximally entangled state [14,29]. Exploiting this feature, one can reformulate Eq. (11) in a basis-independent form:

$$2\max_j|\rho_{j,\bar{j}}| \leq \sum_{i\neq j,\bar{j}} \rho_{i,i}, \quad \forall\ \rho \in \mathcal{D}_N^{2\text{-sep}}. \tag{12}$$

Note that in contrast to the Laskowski-Żukowski condition and the Mermin-type separability inequalities, the fidelity condition does not distinguish biseparability and other forms of k-separability. Indeed, a fully separable state (e.g., $|0^{\otimes N}\rangle$ can already attain the value $F(\rho)=1/2$. Thus the fidelity condition only distinguishes full inseparability (i.e., $k=1$) from other types of separability ($k \geq 2$). However, as will be shown in the next section, violation of the fidelity condition yields a stronger test for full entanglement than violation of the Laskowski-Żukowski condition.

(4) The Dür-Cirac depolarization method [2,4] gives necessary conditions for partial separability under specific bipartite splits. It uses a two-step procedure in which a general state ρ is first depolarized to become a member of a special family of states, called ρ_N, after which this depolarized state is tested for α_2-separability under a bipartite split α_2. If the

depolarized state ρ_N is not separable under α_2, then neither is the original state ρ, but not necessarily vice versa since the depolarization process can decrease inseparability.

The special family of states ρ_N is given by

$$\rho_N = \lambda_0^+|\psi_0^+\rangle\langle\psi_0^+| + \lambda_0^-|\psi_0^-\rangle\langle\psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j(|\psi_j^+\rangle\langle\psi_j^+| + |\psi_j^-\rangle\langle\psi_j^-|), \tag{13}$$

with the so-called orthonormal GHZ-basis $|\psi_j^\pm\rangle = \frac{1}{\sqrt{2}}(|j0\rangle \pm |j'1\rangle)$, where $j=j_1j_2\cdots j_{N-1}$ is in binary notation (i.e., a string of $N-1$ bits), and j' means a bit-flip of j: $j' = j_1'j_2'\cdots j_{N-1}'$, with $j_i'=1,0$ if $j_i=0,1$. The depolarization process does not alter the values of $\lambda_0^\pm = \langle\psi_0^\pm|\rho|\psi_0^\pm\rangle$ and of $\lambda_j = (\langle\psi_j^+|\rho|\psi_j^+\rangle + \langle\psi_j^-|\rho|\psi_j^-\rangle)/2$ of the original state ρ. The values of $j=j_1j_2\cdots j_{N-1}$ can be used to label the various bipartite splits by stipulating that $j=j_1j_2\cdots j_{N-1}$, $j_n=0,(1)$ corresponds to the nth qubit belonging (not belonging) to the same subset as the last qubit. For example, the splits a-(bc), b-(ac), c-(ab) have labels $j=10,01,11$, respectively.

The Dür-Cirac condition [2] says that a state ρ is separable under a specific bipartite split j if

$$|\lambda_0^+ - \lambda_0^-| \leq 2\lambda_j \Leftrightarrow 2|\rho_{1,d}| \leq \rho_{l,l} + \rho_{\bar{l},\bar{l}},$$

$$\forall \rho \in \mathcal{D}_N^j, \quad \bar{l} = d+1-l. \tag{14}$$

For the states (13) this condition is in fact necessary and sufficient. In the right-hand side of the second inequality of Eq. (14) l is determined from j using $\mathrm{Tr}[\rho|\psi_j^+\rangle\langle\psi_j^+| + |\psi_j^-\rangle\langle\psi_j^-|] = \rho_{l,l} + \rho_{\bar{l},\bar{l}}$.

Separability conditions for multipartite splits are constructed from the conditions (14) by means of the partial order $\prec$ of containment. As mentioned above, if a state is α_k-separable, then it is also α_2-separable for all bipartite splits $\alpha_k \prec \alpha_2$. Therefore the conjunction of all α_2-separability conditions must hold for such a state.

Note that if $|\lambda_0^+ - \lambda_0^-| > 2\max_j \lambda_j$, the state is inseparable under all bipartite splits, but this does not imply that it is fully inseparable (cf. [24]). Indeed, this feature also exists for states of the form (13) as the following example shows. Take the following two members of the family (13) for $N=3$: for ρ_3^i we choose $\lambda_0^+=1/2$, $\lambda_0^-=0$, $\lambda_{01}=0$, $\lambda_{10}=1/4$, $\lambda_{11}=0$, and for ρ_3^{ii}: $\lambda_0^+=1/2$, $\lambda_0^-=0$, $\lambda_{01}=0$, $\lambda_{10}=0$, $\lambda_{11}=1/4$. It follows from condition (14) that ρ_3^i is separable under split a-(bc) and inseparable under other splits, while ρ_3^{ii} is separable under the split c-(ab) and inseparable under any other split. Now form a convex mixture of these two states: $\tilde{\rho}_3 = \alpha\rho_3^i + \beta\rho_3^{ii}$ with $\alpha+\beta=1$ and $\alpha,\beta \in (0,1)$. This state $\tilde{\rho}_3$ is still of the form (13), so that we can again apply condition (14) to conclude that $\tilde{\rho}_3$ is not separable under any bipartite split, yet biseparable by construction.

In the next section we give necessary conditions for k-separability and α_k-separability that are stronger than the Laskowski-Żukowski condition (for $k=2,N$), the fidelity condition, and the Dür-Cirac condition.

III. DERIVING PARTIAL SEPARABILITY CONDITIONS

This section presents separability conditions for all levels and classes in the separability hierarchy of N-qubit states. We start with the case of $N=2$, which has been treated more extensively in [30]. We next move on to the slightly more complicated case of three qubits, for which explicit separability conditions are given for each of the ten classes in the separability hierarchy which were depicted in Fig. 1. Finally, the case of N qubits is treated by a straightforward generalization.

A. Two-qubit case: Setting the stage

For two-qubit systems the separability hierarchy is very simple: there is only one possible split, and consequently just one class at each of the two levels $k=1$ and 2, i.e., states are either inseparable (entangled) or separable. Consider a system composed of a pair of qubits in the familiar setting of two distant sites, each receiving one of the two qubits, and where, at each site, a measurement of either of two spin observables is made. We will focus on the special case that these local spin observables are mutually orthogonal. Let $(X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)})$ denote three orthogonal spin observables on qubit a, and $(X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)})$ on qubit b. (The superscript 1 denotes that we are dealing with single-qubit operators.) A familiar choice for the orthogonal triples $\{X^{(1)}, Y^{(1)}, Z^{(1)}\}$ are the Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$; but note that the choice of the two sets need not coincide. We further define $I_{a,b}^{(1)} := \mathbb{1}$. For all single-qubit pure states $|\psi\rangle$ we have

$$\langle X_j^{(1)}\rangle_\psi^2 + \langle Y_j^{(1)}\rangle_\psi^2 + \langle Z_j^{(1)}\rangle_\psi^2 = \langle I_j^{(1)}\rangle_\psi^2, \quad j=a,b, \tag{15}$$

and for mixed states ρ

$$\langle X_j^{(1)}\rangle^2 + \langle Y_j^{(1)}\rangle^2 + \langle Z_j^{(1)}\rangle^2 \leq \langle I_j^{(1)}\rangle^2, \quad j=a,b. \tag{16}$$

We write X_aX_b or even XX, etc. as shorthand for $X_a \otimes X_b$ and $\langle XX\rangle := \mathrm{Tr}[\rho X_a \otimes X_b]$ for the expectation value in a general state ρ, and $\langle XX\rangle_\Psi := \langle\Psi|X_a \otimes X_b|\Psi\rangle$ for the expectation in a pure state $|\Psi\rangle$.

So, let two triples of locally orthogonal observables $\{X_a^{(1)}, Y_a^{(1)}, Z_a^{(1)}\}$ and $\{X_b^{(1)}, Y_b^{(1)}, Z_b^{(1)}\}$ be given, where a,b label the different qubits. We introduce two sets of four two-qubit operators on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, labeled by the subscript $x=0,1$:

$$X_0^{(2)} := \frac{1}{2}(X^{(1)}X^{(1)} - Y^{(1)}Y^{(1)}), \quad X_1^{(2)} := \frac{1}{2}(X^{(1)}X^{(1)} + Y^{(1)}Y^{(1)}),$$

$$Y_0^{(2)} := \frac{1}{2}(Y^{(1)}X^{(1)} + X^{(1)}Y^{(1)}), \quad Y_1^{(2)} := \frac{1}{2}(Y^{(1)}X^{(1)} - X^{(1)}Y^{(1)}),$$

$$Z_0^{(2)} := \frac{1}{2}(Z^{(1)}I^{(1)} + I^{(1)}Z^{(1)}), \quad Z_1^{(2)} := \frac{1}{2}(Z^{(1)}I^{(1)} - I^{(1)}Z^{(1)}),$$

$$I_0^{(2)} := \frac{1}{2}(I^{(1)}I^{(1)} + Z^{(1)}Z^{(1)}), \quad I_1^{(2)} := \frac{1}{2}(I^{(1)}I^{(1)} - Z^{(1)}Z^{(1)}). \tag{17}$$

Here, the superscript label indicates that we are dealing with two-qubit operators. Later on, $X_x^{(2)}$ will sometimes be notated as $X^{(2)}_{x,ab}$, and similarly for $Y^{(2)}_x$, $Z^{(2)}_x$, and $I^{(2)}_x$. This more extensive labeling will prove convenient for the multiqubit generalization. Note that $(X^{(2)}_x)^2=(Y^{(2)}_x)^2=(Z^{(2)}_x)^2=(I^{(2)}_x)^2=I^{(2)}_x$ for $x=0,1$, and that all eight operators mutually anticommute. Furthermore, if the orientations of the two triples are the same, these two sets form representations of the generalized Pauli group, i.e., they have the same commutation relations as the Pauli matrices on $\mathbb{C}^2$, i.e., $[X^{(2)}_x,Y^{(2)}_x]=2iZ^{(2)}_x$, etc. and

$$\langle X^{(2)}_x\rangle^2+\langle Y^{(2)}_x\rangle^2+\langle Z^{(2)}_x\rangle^2\leqslant\langle I^{(2)}_x\rangle^2,\quad x\in\{0,1\},\qquad(18)$$

with equality only for pure states.

Assume for the moment that the two-qubit state is pure and separable. We may thus write $\rho=|\Psi\rangle\langle\Psi|$, where $|\Psi\rangle=|\psi\rangle|\phi\rangle$, to obtain

$$\begin{aligned}\langle X^{(2)}_0\rangle^2_\Psi+\langle Y^{(2)}_0\rangle^2_\Psi&=\langle X^{(2)}_1\rangle^2_\Psi+\langle Y^{(2)}_1\rangle^2_\Psi\\&=\frac{1}{4}(\langle X^{(1)}_a\rangle^2_\psi+\langle Y^{(1)}_a\rangle^2_\psi)(\langle X^{(1)}_b\rangle^2_\phi+\langle Y^{(1)}_b\rangle^2_\phi)\\&=\frac{1}{4}(\langle I^{(1)}_a\rangle-\langle Z^{(1)}_a\rangle^2_\psi)(\langle I^{(1)}_b\rangle-\langle Z^{(1)}_b\rangle^2_\phi)\\&=\langle I^{(2)}_0\rangle^2_\Psi-\langle Z^{(2)}_0\rangle^2_\Psi=\langle I^{(2)}_1\rangle^2_\Psi-\langle Z^{(2)}_1\rangle^2_\Psi.\end{aligned}\qquad(19)$$

This result for pure separable states can be extended to any mixed separable state $\rho\in\mathcal{D}^{2\text{-sep}}_2$ by noting that the density operator of any such state is a convex combination of the density operators for pure product states, i.e., $\rho=\Sigma_j p_j|\Psi_j\rangle\langle\Psi_j|$, with $|\Psi_j\rangle=|\psi_j\rangle|\phi_j\rangle$, $p_j\geqslant 0$, and $\Sigma_j p_j=1$. We may thus write for such states:

$$\begin{aligned}&\sqrt{\langle X^{(2)}_x\rangle^2+\langle Y^{(2)}_x\rangle^2}\\&\quad\leqslant\sum_j p_j\sqrt{\langle X^{(2)}_x\rangle^2_j+\langle X^{(2)}_x\rangle^2_j}\\&\quad=\sum_j p_j\sqrt{\langle I^{(2)}_y\rangle^2_j-\langle Z^{(2)}_y\rangle^2_j}\\&\quad\leqslant\sqrt{\langle I^{(2)}_y\rangle^2-\langle Z^{(2)}_y\rangle^2},\quad\forall\,\rho\in\mathcal{D}^{2\text{-sep}}_2,\quad x,y=0,1.\end{aligned}\qquad(20)$$

Here, $\langle\cdot\rangle_j$ denotes an expectation value in the state $|\Psi_j\rangle$. The first inequality follows because $\sqrt{\langle X^{(2)}_x\rangle^2+\langle Y^{(2)}_x\rangle^2}$ are convex functions of ρ for all x and the second because $\sqrt{\langle I^{(2)}_y\rangle^2-\langle Z^{(2)}_y\rangle^2}$ are concave in ρ for all y. As shown in [30] the right-hand side of this inequality is bounded by 1/2, which follows by considering the equalities of Eq. (19). However, for entangled states [e.g., for the Bell states $|\phi^\pm\rangle=(|00\rangle\pm|11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle=(|01\rangle\pm|10\rangle)/\sqrt{2}$] the left-hand side can attain the value of 1. Hence inequality (20) provides a nontrivial bound for separable states, and thus a criterion for testing entanglement.

In other words, for all separable 2-qubit states one has

$$\begin{aligned}&\max_{x\in\{0,1\}}\langle X^{(2)}_x\rangle^2+\langle Y^{(2)}_x\rangle^2\\&\qquad\leqslant\min_{x\in\{0,1\}}\langle I^{(2)}_x\rangle^2-\langle Z^{(2)}_x\rangle^2\leqslant\frac{1}{4},\quad\forall\,\rho\in\mathcal{D}^{2\text{-sep}}_2.\end{aligned}\qquad(21)$$

In fact, the validity of the inequalities (21) for all orthogonal triples $\{X^{(1)}_a,Y^{(1)}_a,Z^{(1)}_a\}$ and $\{X^{(1)}_b,Y^{(1)}_b,Z^{(1)}_b\}$ provides a necessary and sufficient condition for separability for two-qubit states, pure or mixed. (See [30] for a proof.)

Note that, depending on whether the orientation of the triples of local orthogonal observables is the same or not, the inequalities on the left-hand side of Eq. (21) (leaving out the upperbound 1/4) may be simplified. If we choose the orientations for both parties to be the same, then the interesting separability inequalities in Eq. (21) are $\langle X^{(2)}_0\rangle^2+\langle Y^{(2)}_0\rangle\leqslant\langle I^{(2)}_1\rangle^2-\langle Z^{(2)}_1\rangle^2$ and $\langle X^{(2)}_1\rangle^2+\langle Y^{(2)}_1\rangle\leqslant\langle I^{(2)}_0\rangle^2-\langle Z^{(2)}_0\rangle^2$, whereas the other inequalities in Eq. (21) become trivially true [cf. Eq. (18)]. Choosing the orientations to be different reverses this verdict.

To conclude this section we give an explicit form of the separability inequalities (21) by choosing the Pauli matrices $\{\sigma_x,\sigma_y,\sigma_z\}$ for both triples $\{X^{(1)}_a,Y^{(1)}_a,Z^{(1)}_a\}$ and $\{X^{(1)}_b,Y^{(1)}_b,Z^{(1)}_b\}$. This choice enables us to write the inequalities (21) in terms of the density matrix elements on the standard z-basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, labeled here as $\{|1\rangle,|2\rangle,|3\rangle,|4\rangle\}$. This choice of observables yields $\langle X^{(2)}_0\rangle=2\,\mathrm{Re}\,\rho_{1,4}$, $\langle Y^{(2)}_0\rangle=-2\,\mathrm{Im}\,\rho_{1,4}$, $\langle I^{(2)}_0\rangle=\rho_{1,1}+\rho_{4,4}$, $\langle Z^{(2)}_0\rangle=\rho_{1,1}-\rho_{4,4}$, $\langle X^{(2)}_1\rangle=2\,\mathrm{Re}\,\rho_{2,3}$, $\langle Y^{(2)}_1\rangle=-2\,\mathrm{Im}\,\rho_{2,3}$, $\langle I^{(2)}_1\rangle=\rho_{2,2}+\rho_{3,3}$, $\langle Z^{(2)}_1\rangle=\rho_{2,2}-\rho_{3,3}$. So, in this choice, we can write Eq. (21) as

$$\max\{|\rho_{1,4}|^2,|\rho_{2,3}|^2\}\leqslant\min\{\rho_{1,1}\rho_{4,4},\rho_{2,2}\rho_{3,3}\}\leqslant\frac{1}{16},\quad\rho\in\mathcal{D}^{2\text{-sep}}_2.\qquad(22)$$

In the form (22), it is easy to compare the result to the separability conditions reviewed in Sec. II B. Assume for simplicity that $|\rho_{1,4}|$ is the largest of all the antidiagonal elements $|\rho_{j\bar{j}}|$. Then, for $\rho\in\mathcal{D}^{2\text{-sep}}_2$, and using $\langle M^{(2)}\rangle^2+\langle M'^{(2)}\rangle^2=8(\langle X^{(2)}_0\rangle^2+\langle Y^{(2)}_0\rangle^2)$ the Mermin-type separability inequality (7) becomes $|\rho_{1,4}|^2\leqslant 1/16$, which is equivalent to the Laskowski-Żukowski condition $|\rho_{1,4}|\leqslant 1/4$; the fidelity/Dür-Cirac conditions read: $2|\rho_{1,4}|\leqslant\rho_{2,2}+\rho_{3,3}$; and the condition (22): $|\rho_{1,4}|^2\leqslant\rho_{2,2}\rho_{3,3}$. Using the trivial inequality $(\sqrt{\rho_{22}}-\sqrt{\rho_{33}})^2\geqslant 0\Leftrightarrow 2\sqrt{\rho_{2,2}\rho_{3,3}}\leqslant\rho_{2,2}+\rho_{3,3}$, we can then write the following chain of inequalities:

$$4|\rho_{1,4}|-(\rho_{1,1}+\rho_{4,4})\overset{A}{\leqslant}2|\rho_{1,4}|\overset{\text{sep}}{\leqslant}2\sqrt{\rho_{2,2}\rho_{3,3}}\overset{A}{\leqslant}\rho_{2,2}+\rho_{3,3},\qquad(23)$$

where we used the symbols $\overset{A}{\leqslant}$ and $\overset{\text{sep}}{\leqslant}$ to denote inequalities that hold for all states, and for the separability condition (22), respectively.

The Laskowski-Żukowski condition is then recovered by comparing the first and fourth expressions in this chain, the fidelity/Dür-Cirac conditions by comparing the second and fourth expression, and a new condition—not previously mentioned—can be obtained by comparing the first and third term, whereas condition (22), i.e., the comparison between the second and third expression in Eq. (23), is the strongest inequality in this chain, and thus implies and strengthens all of these other conditions.

B. Three-qubit case

We now derive separability conditions that distinguish the ten classes in the three-qubit classification of Sec. II A by generalizing the method of Sec. III A. To begin with, define four sets of three-qubit observables from the two-qubit operators (17).

$$X_0^{(3)} := \frac{1}{2}(X^{(1)}X_0^{(2)} - Y^{(1)}Y_0^{(2)}), \quad X_1^{(3)} := \frac{1}{2}(X^{(1)}X_0^{(2)} + Y^{(1)}Y_0^{(2)}),$$

$$Y_0^{(3)} := \frac{1}{2}(Y^{(1)}X_0^{(2)} + X^{(1)}Y_0^{(2)}), \quad Y_1^{(3)} := \frac{1}{2}(Y^{(1)}X_0^{(2)} - X^{(1)}Y_0^{(2)}),$$

$$Z_0^{(3)} := \frac{1}{2}(Z^{(1)}I_0^{(2)} + I^{(1)}Z_0^{(2)}), \quad Z_1^{(3)} := \frac{1}{2}(Z^{(1)}I_0^{(2)} - I^{(1)}Z_0^{(2)}),$$

$$I_0^{(3)} := \frac{1}{2}(I^{(1)}I_0^{(2)} + Z^{(1)}Z_0^{(2)}), \quad I_1^{(3)} := \frac{1}{2}(I^{(1)}I_0^{(2)} - Z^{(1)}Z_0^{(2)}),$$

$$X_2^{(3)} := \frac{1}{2}(X^{(1)}X_1^{(2)} - Y^{(1)}Y_1^{(2)}), \quad X_3^{(3)} := \frac{1}{2}(X^{(1)}X_1^{(2)} + Y^{(1)}Y_1^{(2)}),$$

$$Y_2^{(3)} := \frac{1}{2}(Y^{(1)}X_1^{(2)} + X^{(1)}Y_1^{(2)}), \quad Y_3^{(3)} := \frac{1}{2}(Y^{(1)}X_1^{(2)} - X^{(1)}Y_1^{(2)}),$$

$$Z_2^{(3)} := \frac{1}{2}(Z^{(1)}I_1^{(2)} + I^{(1)}Z_1^{(2)}), \quad Z_3^{(3)} := \frac{1}{2}(Z^{(1)}I_1^{(2)} - I^{(1)}Z_1^{(2)}),$$

$$I_2^{(3)} := \frac{1}{2}(I^{(1)}I_1^{(2)} + Z^{(1)}Z_1^{(2)}), \quad I_3^{(3)} := \frac{1}{2}(I^{(1)}I_1^{(2)} - Z^{(1)}Z_1^{(2)}), \quad (24)$$

where $X^{(1)}X_0^{(2)}=X_a^{(1)}\otimes X_{0,bc}^{(2)}$, etc., a,b,c label the three qubits. In analogy to the two-qubit case, we note that all these operators anticommute and that if the orientations of the triples for each qubit are the same, the operators in Eq. (24) yield representations of the generalized Pauli group: $[X_x^{(3)},Y_x^{(3)}]=2iZ_x^{(3)}$, for $x=0,1,2,3$. For convenience, we will indeed assume these orientations to be the same, unless noted otherwise. Choosing orientations differently would yield similar separability conditions, in the same vein as in the previous section. Under this choice we have, for all k,

$$\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 + \langle Z_x^{(3)}\rangle^2 \leqslant \langle I_x^{(3)}\rangle^2, \quad \forall\, \rho \in \mathcal{D}_N^{k\text{-sep}} \quad (25)$$

with equality only for pure states.

We now derive conditions for the different levels and classes of the partial separability classification. Most of the proofs are by straightforward generalization of the method of the previous section and these will be omitted.

Suppose first that the three-qubit state is pure and separable under split a-(bc). From the definitions (24) we obtain

$$\begin{aligned}\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2 &= \frac{1}{4}(\langle X_a^{(1)}\rangle^2 + \langle Y_a^{(1)}\rangle^2)(\langle X_{0,bc}^{(2)}\rangle^2 + \langle Y_{0,bc}^{(2)}\rangle^2)\\ &= \langle X_1^{(3)}\rangle^2 + \langle Y_1^{(3)}\rangle^2\\ &= \langle I_0^{(3)}\rangle^2 - \langle Z_0^{(3)}\rangle^2\\ &= \frac{1}{4}(\langle I_a^{(1)}\rangle^2 - \langle Z_a^{(1)}\rangle^2)(\langle I_{0,bc}^{(2)}\rangle^2 - \langle Z_{0,bc}^{(2)}\rangle^2)\\ &= \langle I_1^{(3)}\rangle^2 - \langle Z_1^{(3)}\rangle^2, \end{aligned} \quad (26)$$

$$\begin{aligned}\langle X_2^{(3)}\rangle^2 + \langle Y_2^{(3)}\rangle^2 &= \frac{1}{4}(\langle X_a^{(1)}\rangle^2 + \langle Y_a^{(1)}\rangle^2)(\langle X_{1,bc}^{(2)}\rangle^2 + \langle Y_{1,bc}^{(2)}\rangle^2)\\ &= \langle X_3^{(3)}\rangle^2 + \langle Y_3^{(3)}\rangle^2\\ &= \langle I_2^{(3)}\rangle^2 - \langle Z_2^{(3)}\rangle^2\\ &= \frac{1}{4}(\langle I_a^{(1)}\rangle^2 - \langle Z_a^{(1)}\rangle^2)(\langle I_{1,bc}^{(2)}\rangle^2 - \langle Z_{1,bc}^{(2)}\rangle^2)\\ &= \langle I_3^{(3)}\rangle^2 - \langle Z_3^{(3)}\rangle^2. \end{aligned} \quad (27)$$

Similarly, for pure states that are separable under split b-(ac), we obtain analogous equalities by interchanging the labels $x=1$ and $x=3$ (denoted as $1\leftrightarrow 3$); and for split c-(ab) by $1\leftrightarrow 2$.

Of course, these equalities hold for pure states only, but by the convex analysis of Sec. III A we obtain from Eqs. (26) and (27) inequalities for all mixed states that are biseparable under the split a-(bc):

$$\left.\begin{aligned}\max_{x\in\{0,1\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min_{x\in\{0,1\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4}\\ \max_{x\in\{2,3\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min_{x\in\{2,3\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4}\end{aligned}\right\}, \quad \forall\, \rho \in \mathcal{D}_3^{a\text{-}(bc)}. \quad (28)$$

For states that are biseparable under split b-(ac) the analogous inequalities with $1\leftrightarrow 3$ hold, i.e.,

$$\left.\begin{aligned}\max_{x\in\{0,3\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min_{x\in\{0,3\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4}\\ \max_{x\in\{1,2\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min_{x\in\{1,2\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4}\end{aligned}\right\}, \quad \forall\, \rho \in \mathcal{D}_3^{b\text{-}(ac)}, \quad (29)$$

and for the split c-(ab) we need to replace $1\leftrightarrow 2$:

$$\begin{array}{l}\max\limits_{x\in\{0,2\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min\limits_{x\in\{0,2\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4} \\ \max\limits_{x\in\{1,3\}} \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant \min\limits_{x\in\{1,3\}} \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \leqslant \frac{1}{4}\end{array}, \quad \forall\, \rho \in \mathcal{D}_3^{c\text{-}(ab)}. \tag{30}$$

A general biseparable state $\rho \in \mathcal{D}_3^{2\text{-sep}}$ is a convex mixture of states that are separable under some bipartite split, i.e., $\rho = p_1\rho_{a\text{-}(bc)} + p_2\rho_{b\text{-}(ac)} + p_3\rho_{c\text{-}(ab)}$ with $\Sigma_{j=1}^3 p_j = 1$. Since $\sqrt{\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2}$ is convex in ρ we get from Eqs. (28)–(30) for such a state:

$$\begin{aligned}\sqrt{\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2} &\leqslant p_1\sqrt{\langle X_0^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}} + \langle Y_0^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}}} + p_2\sqrt{\langle X_0^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}} + \langle Y_0^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}}} + p_3\sqrt{\langle X_0^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}} + \langle Y_0^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}}} \\ &\leqslant p_1\sqrt{\langle I_1^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}} - \langle Z_1^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}}} + p_2\sqrt{\langle I_3^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}} - \langle Z_3^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}}} + p_3\sqrt{\langle I_2^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}} - \langle Z_2^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}}}.\end{aligned} \tag{31}$$

Here $\langle\cdot\rangle_{\rho_{a\text{-}(bc)}}$ means taking the expectation value in the state $\rho_{a\text{-}(bc)}$, etc. Analogous bounds hold for the expressions $\sqrt{\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2}$ for $x=1,2,3$.

From the numerical upper bounds in the conditions (28)–(30) it is easy to obtain a first biseparability condition:

$$\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \leqslant 1/4, \quad \forall\, \rho \in \mathcal{D}_3^{2\text{-sep}}, \quad x \in \{0,1,2,3\}. \tag{32}$$

This is equivalent to the Laskowski-Żukowski condition (3) for $k=2$, as will be shown below. However, a stronger condition can be obtained by noting that $\sqrt{\langle I_y^{(3)}\rangle^2 - \langle Z_y^{(3)}\rangle^2}$ is concave in ρ so that

$$\begin{aligned}p_1\sqrt{\langle I_y^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}} - \langle Z_y^{(3)}\rangle^2_{\rho_{a\text{-}(bc)}}} + p_2\sqrt{\langle I_y^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}} - \langle Z_y^{(3)}\rangle^2_{\rho_{b\text{-}(ac)}}} \\ + p_3\sqrt{\langle I_y^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}} - \langle Z_y^{(3)}\rangle^2_{\rho_{c\text{-}(ab)}}} \leqslant \sqrt{\langle I_y^{(3)}\rangle^2 - \langle Z_y^{(3)}\rangle^2}.\end{aligned} \tag{33}$$

After taking a sum over $y \neq x$ in Eq. (33), the left-hand side of Eq. (33) is larger than the right-hand side of Eq. (31). This yields a stronger condition for biseparability of three-qubit states

$$\sqrt{\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2} \leqslant \sum_{y\neq x} \sqrt{\langle I_y^{(3)}\rangle^2 - \langle Z_y^{(3)}\rangle^2},$$

$$\forall\, \rho \in \mathcal{D}_3^{2\text{-sep}}, \quad x,y \in \{0,1,2,3\}. \tag{34}$$

That Eq. (34) is indeed a stronger than Eq. (32) will be shown below using the density matrix representation of this condition. If one would alter the orientation of the orthogonal triple of observables for a certain qubit, then the right-hand side of Eq. (34) changes by adding either 1, 2, or 3 (modulo 3) to x in the sum on the right-hand side, depending on for which qubit the orientation was changed.

Next, consider the case of a 3-separable state, $\rho \in \mathcal{D}_3^{3\text{-sep}}$. One might then use the fact that this split is contained in all three bipartite splits a-(bc), b-(ac), and c-(ab) to conclude that the inequalities (28)–(30) must hold simultaneously. Thus 3-separable states must obey

$$\max_x\{\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2\} \leqslant \min_x\{\langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2\} \leqslant \frac{1}{4},$$

$$\forall\, \rho \in \mathcal{D}_3^{3\text{-sep}}. \tag{35}$$

However, a more stringent condition holds by virtue of the following equalities for pure 3-separable states:

$$\begin{aligned}\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2 &= \frac{1}{16}(\langle X_a^{(1)}\rangle^2 + \langle Y_a^{(1)}\rangle^2)(\langle X_b^{(1)}\rangle^2 + \langle Y_b^{(1)}\rangle^2) \\ &\quad\times(\langle X_c^{(1)}\rangle^2 + \langle Y_c^{(1)}\rangle^2) = \langle X_1^{(3)}\rangle^2 + \langle Y_1^{(3)}\rangle^2 \\ &= \langle X_2^{(3)}\rangle^2 + \langle Y_2^{(3)}\rangle^2 = \langle X_3^{(3)}\rangle^2 + \langle Y_3^{(3)}\rangle^2,\end{aligned} \tag{36}$$

$$\begin{aligned}\langle I_0^{(3)}\rangle^2 - \langle Z_0^{(3)}\rangle^2 &= \frac{1}{16}(\langle I_a^{(1)}\rangle^2 - \langle Z_a^{(1)}\rangle^2)(\langle I_b^{(1)}\rangle^2 - \langle Z_b^{(1)}\rangle^2)(\langle I_c^{(1)}\rangle^2 \\ &\quad - \langle Z_c^{(1)}\rangle^2) = \langle I_1^{(3)}\rangle^2 - \langle Z_1^{(3)}\rangle^2 = \langle I_2^{(3)}\rangle^2 - \langle Z_2^{(3)}\rangle^2 \\ &= \langle I_3^{(3)}\rangle^2 - \langle Z_3^{(3)}\rangle^2.\end{aligned} \tag{37}$$

From these equalities for pure states it is easy to obtain, by a convexity argument similar to previous cases, an upper bound of 1/16 instead of 1/4 in Eq. (35):

$$\max_x\{\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2\} \leqslant \min_x\{\langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2\} \leqslant \frac{1}{16},$$

$$\forall\, \rho \in \mathcal{D}_3^{3\text{-sep}}. \tag{38}$$

We have thus obtained different conditions for each of the ten classes in the full separability classification of three qubits, summarized in Table I.

Violations of these partial separability conditions give sufficient conditions for particular types of entanglement. For example, if inequality (38) is violated, then the state must be in one of the biseparable classes 2.1–2.8 or in class 1, which implies that the state is at least 2-partite entangled; if Eq. (34) violated it is in class 1 and thus fully inseparable (fully entangled), and so on.

In order to gain further familiarity with the above separability inequalities, we choose the ordinary Pauli matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ for the locally orthogonal observables $\{X^{(1)}, Y^{(1)}, Z^{(1)}\}$, and formulate them in terms of density matrix elements in the standard z basis. Inequalities (28)–(30) now read successively:

$$\begin{array}{l}\max\{|\rho_{1,8}|^2,|\rho_{4,5}|^2\} \leq \min\{\rho_{4,4}\rho_{5,5},\rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{2,7}|^2,|\rho_{3,6}|^2\} \leq \min\{\rho_{2,2}\rho_{7,7},\rho_{3,3}\rho_{6,6}\} \leq 1/16\end{array}, \quad \forall\, \rho \in \mathcal{D}_3^{a\text{-}(bc)}, \tag{39}$$

$$\begin{array}{l}\max\{|\rho_{1,8}|^2,|\rho_{3,6}|^2\} \leq \min\{\rho_{3,3}\rho_{6,6},\rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{2,7}|^2,|\rho_{4,5}|^2\} \leq \min\{\rho_{2,2}\rho_{7,7},\rho_{4,4}\rho_{5,5}\} \leq 1/16\end{array}, \quad \forall\, \rho \in \mathcal{D}_3^{b\text{-}(ac)}, \tag{40}$$

$$\begin{array}{l}\max\{|\rho_{1,8}|^2,|\rho_{2,7}|^2\} \leq \min\{\rho_{2,2}\rho_{7,7},\rho_{1,1}\rho_{8,8}\} \leq 1/16 \\ \max\{|\rho_{3,6}|^2,|\rho_{4,5}|^2\} \leq \min\{\rho_{3,3}\rho_{6,6},\rho_{4,4}\rho_{5,5}\} \leq 1/16\end{array}, \quad \forall\, \rho \in D_3^{c\text{-}(ab)}. \tag{41}$$

For a general biseparable state we can rewrite Eq. (32) as

$$\max\{|\rho_{1,8},|\rho_{2,7}|,|\rho_{3,6}|,|\rho_{4,5}|\,\} \leq 1/4, \quad \forall\, \rho \in \mathcal{D}_3^{\text{2-sep}}. \tag{42}$$

It can easily be seen that this is equivalent to Laskowski-Żukowski's condition (3) for $k=2$. The condition (34) for biseparability yields

$$\begin{array}{l}|\rho_{1,8}| \leq \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{3,3}\rho_{6,6}} + \sqrt{\rho_{4,4}\rho_{5,5}} \\ |\rho_{2,7}| \leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{3,3}\rho_{6,6}} + \sqrt{\rho_{4,4}\rho_{5,5}} \\ |\rho_{3,6}| \leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{4,4}\rho_{5,5}} \\ |\rho_{4,5}| \leq \sqrt{\rho_{1,1}\rho_{8,8}} + \sqrt{\rho_{2,2}\rho_{7,7}} + \sqrt{\rho_{3,3}\rho_{6,6}}\end{array}, \quad \forall\, \rho \in \mathcal{D}_3^{\text{2-sep}}. \tag{43}$$

Finally, condition (35) for general 3-separable states becomes

$$\max\{|\rho_{1,8}|^2,|\rho_{2,7}|^2,|\rho_{3,6}|^2,|\rho_{4,5}|^2\} \leq \min\{\rho_{1,1}\rho_{8,8},\rho_{2,2}\rho_{7,7},\rho_{3,3}\rho_{6,6},\rho_{4,4}\rho_{5,5}\} \leq \frac{1}{64}, \quad \forall\, \rho \in \mathcal{D}_3^{\text{3-sep}}. \tag{44}$$

Note that the separability inequalities (39)–(44) all give bounds on antidiagonal elements in terms of diagonal elements.

We will now show that these bounds improve upon the separability conditions discussed in Sec. II B. We focus on the antidiagonal element $\rho_{1,8}$ (i.e., we suppose that this is the largest antidiagonal matrix element) since this is easiest for comparison. However, the same argument holds for any other antidiagonal matrix element.

The Dür-Cirac conditions in terms of $|\rho_{1,8}|$ read as follows. For partial separability under the split a-(bc): $2|\rho_{1,8}| \leq \rho_{4,4}+\rho_{5,5}$, under the split b-(ac): $2|\rho_{1,8}|\leq \rho_{3,3}+\rho_{6,6}$, and lastly under the split c-(ab): $2|\rho_{1,8}|\leq\rho_{2,2}+\rho_{7,7}$. Next, the Laskowski-Żukowski condition (3) gives for $\rho\in\mathcal{D}_3^{\text{2-sep}}$ that $|\rho_{1,8}|\leq 1/4$ and for $\rho\in\mathcal{D}_3^{\text{3-sep}}$ that $|\rho_{1,8}|\leq 1/8$. The fidelity condition (9) gives that if $\rho\in\mathcal{D}_3^{\text{2-sep}}$ then $2|\rho_{1,8}| \leq\rho_{2,2}+\cdots+\rho_{7,7}$.

TABLE I. Separability conditions for the ten classes in the separability classification of three-qubit states.

Class	Separability conditions
1	(25)
2.1	(34)
2.2	(28)
2.3	(29)
2.4	(30)
2.5	(28) and (29) but not (30)
2.6	(28) and (30) but not (29)
2.7	(29) and (30) but not (28)
2.8	[(28)–(30)] ⇔ (35)
3	(38)

In order to show that all these conditions are implied by our separability conditions, we employ some inequalities which hold for all states ρ: $|\rho_{1,8}|^2\leq\rho_{1,1}\rho_{8,8}$ [this follows from Eq. (25)], and $(\sqrt{\rho_{4,4}}-\sqrt{\rho_{5,5}})^2\geq 0 \Leftrightarrow 2\sqrt{\rho_{4,4}\rho_{5,5}}\leq\rho_{4,4}+\rho_{5,5}$, and similarly $2\sqrt{\rho_{3,3}\rho_{6,6}}\leq\rho_{2,2}+\rho_{6,6}$ and $2\sqrt{\rho_{2,2}\rho_{7,7}}\leq\rho_{2,2}+\rho_{7,7}$. Using these trivial inequalities one easily sees that the conditions (39)–(41) imply the Dür-Cirac conditions for separability under the three bipartite splits. It is also easy to see that the condition for 3-separability (44) strengthens the Laskowski-Żukowski condition (3) for $k=3$. However, it is not so easy to see that Eq. (43) strengthens both the fidelity and Laskowski-Żukowski condition for $k=2$. We will nevertheless show that this is indeed the case.

Let us use the symbols $\overset{A}{\leq}$ and $\overset{\text{2-sep}}{\leq}$ to denote inequalities that hold for all states or for biseparable states, respectively. Combining the above trivial inequalities with condition (43) yields the following sequence of inequalities:

$$4|\rho_{1,8}| - (\rho_{1,1}+\rho_{8,8}) \overset{A}{\leq} 2|\rho_{1,8}| \overset{\text{2-sep}}{\leq} 2\sqrt{\rho_{4,4}\rho_{5,5}} + 2\sqrt{\rho_{3,3}\rho_{6,6}} + 2\sqrt{\rho_{2,2}\rho_{7,7}} \overset{A}{\leq} \rho_{2,2}+\cdots+\rho_{7,7}. \tag{45}$$

The inequality between the second and third expression is Eq. (43). It implies the other inequalities that follow from

Eq. (45). Comparing the first and fourth expression of Eq. (45) one obtains the Laskowski-Żukowski condition (3), while a comparison of the second and fourth yields the fidelity criterion (9). Comparing the first and third term gives a condition which was not previously mentioned. All these are implied by condition (43).

To end this section we show that the separability inequalities for $x=0$ give Mermin-type separability inequalities [28]. Consider the Mermin operator for three qubits:

$$M^{(3)} := X_a^{(1)}X_b^{(1)}Y_c^{(1)} + Y_a^{(1)}X_b^{(1)}X_c^{(1)} + X_a^{(1)}Y_b^{(1)}X_c^{(1)} - Y_a^{(1)}Y_b^{(1)}Y_c^{(1)}, \tag{46}$$

and define $M'^{(3)}$ in the same way, but with all X and Y interchanged. We can now use the identity $16(\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2) = \langle M^{(3)}\rangle^2 + \langle M'^{(3)}\rangle^2$ to obtain from the separability conditions (32) and (38) the following quadratic inequality for k-separability:

$$16(\langle X_0^{(3)}\rangle^2 + \langle Y_0^{(3)}\rangle^2) = \langle M^{(3)}\rangle^2 + \langle M'^{(3)}\rangle^2 \leq 64\left(\frac{1}{4}\right)^k, \quad \forall \rho \in \mathcal{D}_3^{k\text{-sep}}. \tag{47}$$

Of course, a similar bound holds when $\langle X_0\rangle^2 + \langle Y_0\rangle^2$ in the left-hand side is replaced by $\langle X_x\rangle^2 + \langle Y_x\rangle^2$ for $x=1,2,3$. This reproduces, for $N=3$, the result (7) of Ref. [1]. From the density matrix representation, we see that these Mermin-type separability conditions are in fact equivalent to the Laskowski-Żukowski condition (3). Note that these conditions do not distinguish the different classes within level $k=2$, as was the case in Eqs. (39)–(41).

C. N-qubit case

In this section we generalize the analysis of the previous section to N qubits to obtain conditions for k-separability and α_k-separability. The proofs are analogous to the previous cases, and will be omitted. Explicit conditions for k-separability are presented for all levels $k=1,\ldots,N$. Further, we give a recursive procedure to derive α_k-separability conditions for each k-partite split α_k at all level k. From these, one can easily construct the conditions that distinguish all the classes in N-partite separability classification by enumerating all possible logical combinations of separability or inseparability under each of these splits at a given level. We will, however, not attempt to write down these latter conditions explicitly since the number of classes grows exponentially with the number of qubits. We start by considering bipartite splits, and biseparable states (level $k=2$), and then move upwards to obtain separability conditions for splits on higher levels.

We define $2^{(N-1)}$ sets of four observables $\{X_x^{(N)}, Y_x^{(N)}, Z_x^{(N)}, I_x^{(N)}\}$, with $x \in \{0,1,\ldots,2^{(N-1)}-1\}$ recursively from the $(N-1)$-qubit observables:

$$\begin{aligned}
X_y^{(N)} &:= \frac{1}{2}(X^{(1)} \otimes X_{y/2}^{(N-1)} - Y^{(1)} \otimes Y_{y/2}^{(N-1)}), & X_{y+1}^{(N)} &:= \frac{1}{2}(X^{(1)} \otimes X_{y/2}^{(N-1)} + Y^{(1)} \otimes Y_{y/2}^{(N-1)}),\\
Y_y^{(N)} &:= \frac{1}{2}(Y^{(1)} \otimes X_{y/2}^{(N-1)} + X^{(1)} \otimes Y_{y/2}^{(N-1)}), & Y_{y+1}^{(N)} &:= \frac{1}{2}(Y^{(1)} \otimes X_{y/2}^{(N-1)} - X^{(1)} \otimes Y_{y/2}^{(N-1)}),\\
Z_y^{(N)} &:= \frac{1}{2}(Z^{(1)} \otimes I_{y/2}^{(N-1)} + I^{(1)} \otimes Z_{y/2}^{(N-1)}), & Z_{y+1}^{(N)} &:= \frac{1}{2}(Z^{(1)} \otimes I_{y/2}^{(N-1)} - I^{(1)} \otimes Z_{y/2}^{(N-1)}),\\
I_y^{(N)} &:= \frac{1}{2}(I^{(1)} \otimes I_{y/2}^{(N-1)} + Z^{(1)} \otimes Z_{y/2}^{(N-1)}), & I_{y+1}^{(N)} &:= \frac{1}{2}(I^{(1)} \otimes I_{y/2}^{(N-1)} - Z^{(1)} \otimes Z_{y/2}^{(N-1)}),
\end{aligned} \tag{48}$$

with y even, i.e., $y \in \{0,2,4,\ldots\}$. Analogous relations between these observables hold as those between the observables (17) and (24). In particular, if the orientations of each triple of local orthogonal observables is the same, these sets form representations of the generalized Pauli group, and every N-qubit state obeys $\langle X_x^{(N)}\rangle^2 + \langle Y_x^{(N)}\rangle^2 \leq \langle I_x^{(N)}\rangle^2 - \langle Z_x^{(N)}\rangle^2$, with equality only for pure states.

1. Biseparability

Consider a state that is separable under some bipartite split α_2 of the N qubits. For each such split we get $2^{(N-1)}$ separability inequalities in terms of the sets $\{X_x^{(N)}, Y_x^{(N)}, Z_x^{(N)}, I_x^{(N)}\}$ labeled by $x \in \{0,1,\ldots,2^{(N-1)}-1\}$. These separability inequalities provide necessary conditions for the N-qubit state to be separable under the split under consideration. In order to find these inequalities, we first determine the N-qubit analogs of the three-qubit pure state equalities (26) and (27) corresponding to this bipartite split. We have not found a generic expression that lists them all for each possible split and all x. However, for the split where the first qubit is separated from the $(N-1)$ other qubits, i.e., $\alpha_2 = a\text{-}(bc\cdots n)$ a generic form can be given:

TABLE II. Solution sets for the seven different bipartite splits of four qubits.

Split α_2	a-(bcd)	b-(acd)	c-(abd)	d-(abc)	(ab)-(cd)	(ac)-(bd)	(ad)-(bc)
$z_1^{\alpha_2}$	{0,1}	{0,3}	{0,6}	{0,4}	{0,2}	{0,7}	{0,5}
$z_2^{\alpha_2}$	{2,3}	{1,2}	{1,7}	{1,5}	{1,3}	{1,6}	{1,4}
$z_3^{\alpha_2}$	{4,5}	{5,6}	{2,4}	{2,6}	{4,6}	{2,5}	{2,7}
$z_4^{\alpha_2}$	{6,7}	{4,7}	{3,5}	{3,7}	{5,7}	{3,4}	{3,6}

$$\begin{aligned}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2 &= \frac{1}{4}(\langle X_a^{(1)}\rangle^2+\langle Y_a^{(1)}\rangle^2)(\langle X_{x/2}^{(N-1)}\rangle^2+\langle Y_{x/2}^{(N-1)}\rangle^2)\\ &=\langle X_{x+1}^{(N)}\rangle^2+\langle Y_{x+1}^{(N)}\rangle^2=\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\\ &=\frac{1}{4}(\langle I_a^{(1)}\rangle^2-\langle Z_a^{(1)}\rangle^2)(\langle I_{x/2}^{(N-1)}\rangle^2-\langle Z_{x/2}^{(N-1)}\rangle^2)\\ &=\langle I_{x+1}^{(N)}\rangle^2-\langle Z_{x+1}^{(N)}\rangle^2, \end{aligned}\tag{49}$$

where, without loss of generality, x is chosen to be even, i.e., $x\in\{0,2,4,\dots\}$. For other bipartite splits the sets of observables labeled by x are permuted, in a way depending on the particular split.

For example, for $N=4$ where $x\in\{0,1,\dots,7\}$ the equalities (49) give the result for the split a-(bcd). The corresponding equalities for other bipartite splits are obtained by the following permutations of x: for split b-(acd): $1\leftrightarrow 3$ and $5\leftrightarrow 7$; for split c-(abd): $1\leftrightarrow 6$ and $3\leftrightarrow 4$; and for split d-(abc): $1\leftrightarrow 4$ and $3\leftrightarrow 6$. For the split (ab)-(cd): $1\leftrightarrow 2$ and $5\leftrightarrow 6$; for (ac)-(bd): $1\leftrightarrow 7$ and $3\leftrightarrow 5$; and lastly, for (ad)-(bc): $1\leftrightarrow 5$ and $3\leftrightarrow 7$.

For mixed states that are separable under a given bipartite split the equalities (49) (and their analogs obtained via suitable permutations) become inequalities. We again state them for the split a-($bc\cdots n$):

$$\max\left\{\begin{matrix}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\\ \langle X_{x+1}^{(N)}\rangle^2+\langle Y_{x+1}^{(N)}\rangle^2\end{matrix}\right\}\leqslant\min\left\{\begin{matrix}\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\\ \langle I_{x+1}^{(N)}\rangle^2-\langle Z_{x+1}^{(N)}\rangle^2\end{matrix}\right\}\leqslant\frac{1}{4},\quad\forall\,\rho\in\mathcal{D}_N^{a\text{-}(bc\cdots n)}\quad\text{with } x\in\{0,2,4,\dots\}.\tag{50}$$

The proof of Eq. (50) is a straightforward generalization of the convex analysis in Sec. III A. Again, for the other bipartite splits, the labels x are permuted in a way depending on the particular split.

For a general biseparable state $\rho\in\mathcal{D}_N^{2\text{-sep}}$, we thus obtain the following biseparability conditions:

$$\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant 1/4,\quad\forall\,x,\quad\forall\,\rho\in\mathcal{D}_N^{2\text{-sep}},\tag{51}$$

which is equivalent to the Laskowski-Żukowski condition for $k=2$ (as will be shown below); and just as in the three-qubit case, we also obtain a stronger condition

$$\sqrt{\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2}\leqslant\sum_{y\neq x}\sqrt{\langle I_y^{(N)}\rangle^2-\langle Z_y^{(N)}\rangle^2},\quad\forall\,\rho\in\mathcal{D}_N^{2\text{-sep}},\quad\text{with } x,y=0,1,\dots,2^{(N-1)}-1.\tag{52}$$

Violation of this inequality is a sufficient condition for full inseparability, i.e., for full N-partite entanglement.

The inequalities (52) are stronger than the fidelity criterion (9) and the Laskowski-Żukowski criterion (3) for $k=2$, and inequalities (50) are stronger than the Dür-Cirac condition (14) for separability under bipartite splits. This will be shown below in Sec. III C 3.

2. Partial separability criteria for levels $2<k\leqslant N$

For levels $k>2$ we sketch a procedure to find α_{k+1}-separability inequalities recursively from inequalities at the preceding level. Suppose that at level k the inequalities are given for separability under each k-partite split α_k of the N qubits, and that these α_k-separability inequalities take the form

$$\max_{x\in z_i^{\alpha_k}}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\min_{x\in z_i^{\alpha_k}}\langle I_x\rangle^2-\langle Z_x\rangle^2\leqslant\frac{1}{4^{(k-1)}},\quad\forall\,\rho\in\mathcal{D}_N^{\alpha_k},\quad i\in\{1,2,\dots,2^{(N-k)}\},\tag{53}$$

where $z_i^{\alpha_k}$ denote "solution sets" for the specific k-partite split α_k. For example, in the case of three qubits, the solution sets for the bipartite split a-(bc) are $z_1^{a\text{-}(bc)}=\{0,1\}$ and $z_2^{a\text{-}(bc)}=\{2,3\}$, as can be seen from Eq. (28). The solution sets for other bipartite splits can be read off Eqs. (29) and (30) so as to give $z_1^{b\text{-}(ac)}=\{0,3\}$, $z_2^{b\text{-}(ac)}=\{1,2\}$, and $z_1^{c\text{-}(ab)}=\{0,2\}$, $z_2^{c\text{-}(ab)}=\{1,3\}$; and for future purposes we list them for the case of four qubits in Table II above. These were obtained by determining Eq. (50) for $N=4$ and for all bipartite splits α_2.

Now move one level higher and consider a given $(k+1)$-partite split $\alpha_{(k+1)}$. This split is contained in a total number of $\binom{k+1}{2}=k(k+1)/2$ k-partite splits α_k. Call the collection of these k-partite splits $\mathcal{S}_{\alpha_{(k+1)}}$. We then obtain preliminary separability

TABLE III. Solution sets for the six different 3-partite splits of four qubits.

Split α_3	a-b-(cd)	(ab)-c-d	a-b-(cd)	(ac)-b-d	(ad)-b-c	(bd)-a-c
$z_1^{\alpha_3}$	{0,1,2,3}	{0,2,4,6}	{0,1,4,5}	{0,3,4,7}	{0,3,5,6}	{0,1,6,7}
$z_2^{\alpha_3}$	{4,5,6,7}	{1,3,5,7}	{2,3,6,7}	{1,2,5,6}	{1,2,4,7}	{2,3,4,5}

inequalities for the split α_{k+1} from the conjunction of all separability inequalities for the splits α_k in the set $\mathcal{S}_{\alpha_{(k+1)}}$. To be specific, this yields

$$\max_{\alpha_k\in\mathcal{S}_{\alpha_{k+1}}}\max_{x\in z_i^{\alpha_k}}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\min_{\alpha_k\in\mathcal{S}_{\alpha_{(k+1)}}}\min_{x\in z_i^{\alpha_k}}\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\leqslant\frac{1}{4^{k-1}},\quad\forall\rho\in\mathcal{D}_N^{\alpha_{(k+1)}}. \tag{54}$$

This may be written more compactly as

$$\max_{x\in z_i^{\alpha_{k+1}}}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\min_{x\in z_i^{\alpha_{k+1}}}\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\leqslant\frac{1}{4^{k-1}},\quad\forall\rho\in\mathcal{D}_N^{\alpha_{(k+1)}},\quad i\in\{1,2,\ldots,2^{(N-k-1)}\}. \tag{55}$$

(In fact, this can be regarded as an implicit definition of the solution sets $z_i^{\alpha_{k+1}}$.) More importantly, by an argument similar to that leading from Eq. (35) to Eq. (38) one finds a stronger numerical bound in the utmost right-hand side of these inequalities, namely 4^{-k} instead of $4^{-(k-1)}$. Thus the final result is

$$\max_{x\in z_i^{\alpha_{k+1}}}\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\min_{x\in z_i^{\alpha_{k+1}}}\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\leqslant\frac{1}{4^{k}},\quad\forall\rho\in\mathcal{D}_N^{\alpha_{(k+1)}},\quad i\in\{1,2,\ldots,2^{(N-k-1)}\}. \tag{56}$$

This shows that the α_k-separability inequalities indeed take the same form as Eq. (53) at all levels.

As an example of this recursive procedure, take $N=4$, set $k=3$, and choose the split a-b-(cd). This split is contained in three 2-partite splits a-(bcd), b-(acd) and (ab)-(cd). Using Eq. (54) and the first, second, and fifth column of Table II one obtains the following two solutions sets for the split a-b-(cd): $z_1^{a\text{-}b\text{-}(cd)}=\{0,1,2,3\}$ and $z_2^{a\text{-}b\text{-}(cd)}=\{4,5,6,7\}$. This leads to the separability inequalities

$$\left.\begin{array}{l}\max\limits_{x\in\{0,1,2,3\}}\langle X_x^{(4)}\rangle^2+\langle Y_x^{(4)}\rangle^2\leqslant\min\limits_{x\in\{0,1,2,3\}}\langle I_x^{(4)}\rangle^2-\langle Z_x^{(4)}\rangle^2\leqslant\dfrac{1}{16}\\ \max\limits_{x\in\{4,5,6,7\}}\langle X_x^{(4)}\rangle^2+\langle Y_x^{(4)}\rangle^2\leqslant\min\limits_{x\in\{4,5,6,7\}}\langle I_x^{(4)}\rangle^2-\langle Z_x^{(4)}\rangle^2\leqslant\dfrac{1}{16}\end{array}\right.,\quad\forall\rho\in\mathcal{D}_4^{a\text{-}b\text{-}(cd)}. \tag{57}$$

For other 3-partite splits the inequalities can be obtained in a similar way so as to give Table III above.

As a special case, we mention the result for full separability, i.e., for $k=N$. There is only one N-partite split, namely where all qubits end up in a different set. Further, there is only one solution set $z_i^{\alpha_N}$ and it contains all $x\in\{0,1,\ldots,2^{(N-1)}-1\}$. States ρ that are separable under this split thus obey

$$\max_x\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\min_x\langle I_x^{(N)}\rangle^2-\langle Z_x^{(N)}\rangle^2\leqslant\frac{1}{4^{(N-1)}},\quad\forall\rho\in\mathcal{D}_N^{N\text{-sep}}. \tag{58}$$

Violation of this inequality is a sufficient condition for some entanglement to be present in the N-qubit state. The condition (58) strengthens the Laskowski-Żukowski condition (3) for $k=N$ (to be shown below).

For an N-qubit k-separable state $\rho\in\mathcal{D}_N^{k\text{-sep}}$, i.e., a state that is a convex mixture of states that are separable under some k-partite split, we obtain from Eq. (56) the following k-separability conditions:

$$\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2\leqslant\frac{1}{4^{(k-1)}},\quad\forall x,\quad\forall\rho\in\mathcal{D}_N^{k\text{-sep}}, \tag{59}$$

which is equivalent to the Laskowski-Żukowski condition (3) for all N and k (this will be shown below using the density matrix formulation of these conditions). However, in analogy to Eq. (34) we also obtain the stronger condition:

$$\sqrt{\langle X_x^{(N)}\rangle^2+\langle Y_x^{(N)}\rangle^2}\leqslant\min_l\sum_{y\in\mathcal{T}_{k,l}^{N,x}}\sqrt{\langle I_y^{(N)}\rangle^2-\langle Z_y^{(N)}\rangle^2},\quad\forall\rho\in\mathcal{D}_N^{k\text{-sep}}, \tag{60}$$

where, for given N, k, and x, $\mathcal{T}_{k,l}^{N,x}$ denotes a tuple of values of $y \neq x$, each one being picked from each of the solutions sets $z_i^{\alpha_k}$ that contain x, where α_k ranges over all the k-partite splits of the N qubits. In general, there will be many ways of picking such values, and we use l as an index to label such tuples.

For example, in the case $N=3$, there are a total of six solution sets (two for each of the three bipartite splits): {0, 1}, {2, 3}, {0, 2}, {1, 3}, {0, 3}, {1, 2}. If we set $x=0$ and pick a member different from 0 from each of those sets that contain 0, we find $\mathcal{T}_{2,1}^{3}=\{1,2,3\}$. This is in fact the only such choice and thus $l=1$. Thus in this example condition (60) reproduces the result (34).

As a more complicated example, take $N=4$, $k=3$, and choose again $x=0$. In this case there are six 3-partite splits each of which has two solution sets, as given in Table III. The solution sets that contain 0 are all on the top row of this table. There are now many ways of constructing a tuple by picking elements that differ from 0 from each of these sets, for example, $\mathcal{T}_{3,1}^{4,0}=\{1,2,1,3,3,1\}$, $\mathcal{T}_{3,2}^{4,0}=\{1,2,1,3,3,6\}$, etc. In this case one has to take a minimum in Eq. (60) over all these $l=1,\ldots,3^6$ tuples.

For $k=2$, condition (60) reduces to Eq. (52) and for $k=N$ to Eq. (58). For these values of k, the condition is stronger than Eq. (59) (see the next section). For $k\neq 2$, N, this is still an open question.

To conclude this section, let us recapitulate. We have found separability conditions in terms of local orthogonal observables for each of the N parties that are necessary for k-separability and for separability under splits α_k at each level on the hierarchic separability classification. Violations of these separability conditions give sufficient criteria for k-separable entanglement and m-partite entanglement with $\lceil N/k\rceil \leq m \leq N-k+1$. The separability conditions are stronger than the Dür-Cirac condition for separability under specific splits, and stronger than the fidelity condition and the Laskowski-Żukowski condition for biseparability. The latter condition is also strengthened for $k=N$. These implications are shown in the next section.

3. The conditions in terms of matrix elements

Choosing the Pauli matrices $\{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}\}$ as local orthogonal observables, with the same orientation at each qubit, allows one to formulate the separability conditions in terms of the density matrix elements $\rho_{i,j}$ on the standard z basis [31]. For these choices we obtain:

$$X_0^{(N)} = |0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N}, \quad \langle X_0^{(N)}\rangle = 2\,\mathrm{Re}\,\rho_{1,d},$$

$$Y_0^{(N)} = -i|0\rangle\langle 1|^{\otimes N} + i|1\rangle\langle 0|^{\otimes N}, \quad \langle Y_0^{(N)}\rangle = -2\,\mathrm{Im}\,\rho_{1,d},$$

$$I_0^{(N)} = |0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N}, \quad \langle I_0^{(N)}\rangle = \rho_{1,1} + \rho_{d,d},$$

$$Z_0^{(N)} = |0\rangle\langle 0|^{\otimes N} - |1\rangle\langle 1|^{\otimes N}, \quad \langle Z_0^{(N)}\rangle = \rho_{1,1} - \rho_{d,d}, \tag{61}$$

where $d=2^N$. Analogous relations hold for $X_x^{(N)}$, $Y_x^{(N)}$, $Z_x^{(N)}$, $I_x^{(N)}$ for $x\neq 0$.

Let us treat the case $N=4$ in detail. First, consider the level $k=2$. Biseparability under the split a-(bcd) gives the following inequalities for the antidiagonal matrix elements:

$$\left.\begin{aligned}
&\max\{|\rho_{1,16}|^2,|\rho_{8,9}|^2\} \leq \min\{\rho_{1,1}\rho_{16,16},\rho_{8,8}\rho_{9,9}\} \leq 1/16\\
&\max\{|\rho_{2,15}|^2,|\rho_{7,10}|^2\} \leq \min\{\rho_{2,2}\rho_{15,15},\rho_{7,7}\rho_{10,10}\} \leq 1/16\\
&\max\{|\rho_{3,14}|^2,|\rho_{6,11}|^2\} \leq \min\{\rho_{3,3}\rho_{14,14},\rho_{6,6}\rho_{11,11}\} \leq 1/16\\
&\max\{|\rho_{5,12}|^2,|\rho_{4,13}|^2\} \leq \min\{\rho_{5,5}\rho_{12,12},\rho_{4,4}\rho_{13,13}\} \leq 1/16
\end{aligned}\right\}, \quad \forall\,\rho \in \mathcal{D}_4^{a\text{-}(bcd)}. \tag{62}$$

The analogous inequalities for separability under other bipartite splits are obtained by suitable permutations on the labels. Indeed, for split b-(acd) labels 8 and 5, 9 and 12, 2 and 3, and 5 and 14 are permuted, which we denote as $(8,9,2,15)\leftrightarrow(5,12,3,14)$; for split c-(abd): $(8,9,2,15)\leftrightarrow(3,14,5,12)$; for split d-(abc): $(8,9,3,14)\leftrightarrow(2,15,5,12)$; for the split (ab)-(cd): $(8,9,3,14)\leftrightarrow(4,13,7,10)$; for (ac)-(bd): $(8,9,5,12)\leftrightarrow(6,11,7,10)$; and lastly, for the split (ad)-(bc): $(8,9,5,12)\leftrightarrow(7,10,6,11)$. For a general biseparable state we obtain

$$|\rho_{1,16}| \leq \sqrt{\rho_{2,2}\rho_{15,15}} + \sqrt{\rho_{3,3}\rho_{14,14}} + \cdots + \sqrt{\rho_{8,8}\rho_{9,9}}, \quad \forall\,\rho \in \mathcal{D}_4^{2\text{-sep}}, \tag{63}$$

and analogous for the other antidiagonal elements.

Next, consider one level higher, i.e., $k=3$. There are six different 3-partite splits for a system consisting of four qubits. For separability under each such split a different set of inequalities can be obtained from Eq. (54). To be more precise, such a set consists of the conjunction of all the separability inequalities for the bipartite splits at level $k=2$ this particular 3-partite split is contained in. For $N=4$ each 3-partite split is contained in three bipartite splits. For example, for separability under split a-b-(cd) we obtain

$$\left.\begin{aligned}
&\max\{|\rho_{1,16}|^2,|\rho_{8,9}|^2,|\rho_{4,13}|^2,|\rho_{5,12}|^2\} \leq \min\{\rho_{1,1}\rho_{16,16},\rho_{8,8}\rho_{9,9},\rho_{4,4}\rho_{13,13},\rho_{5,5}\rho_{12,12}\} \leq 1/64.\\
&\max\{|\rho_{2,15}|^2,|\rho_{3,14}|^2,|\rho_{6,11}|^2,|\rho_{7,10}|^2\} \leq \min\{\rho_{2,2}\rho_{15,15},\rho_{3,3}\rho_{14,14},\rho_{6,6}\rho_{11,11},\rho_{7,7}\rho_{10,10}\} \leq 1/64
\end{aligned}\right\}, \quad \forall\,\rho \in \mathcal{D}_4^{a\text{-}b\text{-}(cd)}. \tag{64}$$

This is the density matrix formulation of Eq. (57).

A general 3-separable state $\rho \in \mathcal{D}_4^{3\text{-sep}}$ is a convex mixture of states that each are separable under some such 3-partite split. The separability condition follows from Eq. (60):

$$|\rho_{1,16}| \leq \min_l \left(\sum_{j \in \tilde{T}^{4,0}_{3,l}} \sqrt{\rho_{j,j}\rho_{17-j,17-j}} \right), \quad \forall \rho \in \mathcal{D}_4^{3\text{-sep}}, \tag{65}$$

where $\tilde{T}^{4,0}_{3,l}$ is the tuple of indices j in {1,16} that label the antidiagonal density matrix elements $\rho_{j,17-j}$ corresponding to the density matrix formulation of the set of operators $\langle X_y^{(4)}\rangle^2+\langle Y_y^{(4)}\rangle$ with y determined by $\mathcal{T}^{4,0}_{3,l}$. Here we have used that the antidiagonal element $\rho_{1,16}$ corresponds to $\langle X_0^{(4)}\rangle^2+\langle Y_0^{(4)}\rangle^2$. For $N=4$, $k=3$ there are six possible splits, so for each l, j is picked from a total of six sets. For the case under consideration the sets are {1, 4, 5, 8}, {1, 2, 3, 4}, {1, 3, 5, 7}, {1, 2, 5, 6}, {1, 2, 7, 8}, and {1, 3, 6, 8}. For each l one chooses a tuple of values of j where one value is picked from each of these six sets, except for the value 1 which is excluded. Analogous inequalities are obtained for the other antidiagonal matrix elements.

Finally for full separability ($k=4$) we get

$$\max\{|\rho_{1,16}|^2,|\rho_{2,15}|^2, \ldots ,|\rho_{8,9}|^2\} \leq \min\{\rho_{1,1}\rho_{16,16},\rho_{2,2}\rho_{15,15}, \ldots ,\rho_{8,8}\rho_{9,9}\} \leq 1/256, \quad \forall \rho \in \mathcal{D}_4^{4\text{-sep}}. \tag{66}$$

For general N, it is easy to see that Eq. (51) yields the Laskowski-Żukowski condition (3). It is instructive to look at the extremes of biseparability and full separability, since for them explicit forms can be given. For $k=2$ condition (52) reads

$$|\rho_{l,\bar{l}}| \leq \sum_{n\neq l,\bar{l}} \sqrt{\rho_{n,n}\rho_{\bar{n},\bar{n}}}/2, \quad \forall \rho \in \mathcal{D}_N^{2\text{-sep}}, \quad \text{where } \bar{l}=d+1-l, \quad \bar{n}=d+1-n, \quad l,n \in \{1, \ldots ,d\}. \tag{67}$$

For $k=N$, we can reformulate condition (58) as

$$\max\{|\rho_{1,d}|^2,|\rho_{2,d-1}|^2 \cdots \} \leq \min\{\rho_{1,1}\rho_{d,d},\rho_{2,2}\rho_{d-1,d-1}, \ldots \} \leq 1/4^N, \quad \forall \rho \in \mathcal{D}_N^{N\text{-sep}}. \tag{68}$$

It is easily seen that the condition (68) is stronger than the Laskowski-Żukowski condition (3) for this case.

Again, these inequalities give bounds on antidiagonal matrix elements in terms of diagonal ones on the z basis. These density matrix representations depend on the choice of the Pauli matrices as the local observables. However, every other triple of locally orthogonal observables with the same orientation can be obtained from the Pauli matrices by suitable local basis transformations, and therefore this matrix representation does not lose generality. Choosing different orientations of the triples one obtains the corresponding inequalities by suitable permutations of antidiagonal matrix elements.

We will now show that Eq. (67) is indeed stronger than the fidelity condition (9) and the Laskowski-Żukowski condition (3) for $k=2$ by following the same analysis as in the three-qubit case. We again assume, for convenience, that the antidiagonal element $\rho_{1,d}$ is the largest of all antidiagonal elements. Using some inequalities that hold for all states together with the condition (67) for biseparability we get the following sequence of inequalities for $\rho_{1,d}$:

$$4|\rho_{1,d}| - (\rho_{1,1} + \rho_{d,d}) \overset{A}{\leq} 2|\rho_{1,d}| \overset{2\text{sep}}{\leq} 2\sqrt{\rho_{2,2}\rho_{d-1,d-1}} + \cdots + 2\sqrt{\rho_{d/2,d/2}\rho_{d/2+1,d/2+1}} \overset{A}{\leq} \rho_{22} + \cdots + \rho_{d-1,d-1}. \tag{69}$$

The inequality in the middle is Eq. (67). It implies all other inequalities in the sequence (69). The inequality between the first and fourth term yields the Laskowski-Żukowski condition for $k=2$, and between the second and fourth gives the fidelity criterion in the formulation (11). One also sees that the fidelity criterion is stronger than the Laskowski-Żukowski condition for $k=2$.

We finally discuss two examples showing that the biseparability condition (67) is stronger in detecting full entanglement than other methods. First, consider the family of N-qubit states

$$\rho'_N = \lambda_0^+|\psi_0^+\rangle\langle\psi_0^+| + \lambda_0^-|\psi_0^-\rangle\langle\psi_0^-| + \sum_{j=1}^{2^{N-1}-1} \lambda_j(|\psi_k^+\rangle + |\psi_j^-\rangle)(\langle\psi_j^+| + \langle\psi_j^-|). \tag{70}$$

The states (70) violate Eq. (67) for all $|\lambda_0^+-\lambda_0^-|\neq 0$ and are thus detected as fully entangled by that condition. In that case they are also inseparable under any split. The fidelity criterion (11), however, detects these states as fully entangled only for $|\lambda_0^+-\lambda_0^-|\geq \Sigma_j\lambda_j$. Violation of Eq. (67) thus allows for detecting more states of the form ρ'_N as fully entangled than violation of the fidelity criterion. Further, the Dür-Cirac criteria detects these states as inseparable under any split for $|\lambda_0^+-\lambda_0^-|> 2\lambda_j$, $\forall j$, which includes less states than a violation of Eq. (67). This generalizes the observation of Ref. [32] from two qubits to the N-qubit case.

Second, consider the N-qubit GHZ-like states $|\theta\rangle = \cos\theta|0\rangle^{\otimes N}+\sin\theta|1\rangle^{\otimes N}$. We can easily read off from the density matrix $|\theta\rangle\langle\theta|$ that the far off-antidiagonal matrix elements $\rho_{1,d}=\rho_{d,1}$ is equal to $\cos\theta\sin\theta$ and that the diagonal matrix elements $\rho_{2,2}, \ldots ,\rho_{d-1,d-1}$ are all equal to zero. Using Eq. (67) we see that these states are fully N-partite entangled for $\rho_{1,d}=\cos\theta\sin\theta\neq 0$, i.e., for all $\theta\neq 0,\pi/2 \pmod{\pi}$. Thus all fully entangled states of this form are detected by condition (67), including those not detectable by any standard multipartite Bell inequality [33].

4. Relationship to Mermin-type inequalities for partial separability and LHV models

We will now show that the separability inequalities of the previous section imply already known Mermin-type inequalities [28] for partial separability. Using the identity

$2^{(N+1)}(\langle X_0^{(N)}\rangle^2+\langle Y_0^{(N)}\rangle^2)=\langle M^{(N)}\rangle^2+\langle M'^{(N)}\rangle^2$ for the Mermin operators (6) together with the upper bound for the separability inequality of Eq. (59) for $x=0$ gives the following sharp quadratic inequality:

$$\langle M^{(N)}\rangle^2+\langle M'^{(N)}\rangle^2 \leq 2^{(N+3)}\left(\frac{1}{4}\right)^k, \quad \forall\, \rho \in \mathcal{D}_N^{k\text{-sep}}, \quad (71)$$

which reproduces the result (7) found by [1]. Since Eq. (51) is equivalent to Eq. (3) we see that the Mermin-type separability condition is in fact one of the Laskowski-Żukowski conditions written in terms of local observables X and Y.

As a special case we consider a split of the form $\{1\},\ldots,\{\kappa\},\{\kappa+1,\ldots,n\}$. Any state that is separable under this split is $(\kappa+1)$-separable so we get the condition $\langle M^{(N)}\rangle^2+\langle M'^{(N)}\rangle^2 \leq 2^{(N-2\kappa+1)}$, and hence $|\langle M^{(N)}\rangle| \leq 2^{(N-2\kappa+1)/2}$. This strengthens the result of Gisin and Bechmann-Pasquinucci [9] by a factor $2^{\kappa/2}$ for these specific Mermin operators (6).

As another special case of the inequalities (71), consider $k=N$. In this case, the inequalities express a condition for full separability of ρ. These inequalities are maximally violated by fully entangled states by an exponentially increasing factor of 2^{N-1}, since the maximal value of $|\langle M^{(N)}\rangle|$ for any quantum state ρ is $2^{(N+1)/2}$ [34]. Furthermore, LHV models violate them also by an exponentially increasing factor of $2^{(N-1)/2}$, since for all N, LHV models allow a maximal value for $|\langle M^{(N)}\rangle|$ of 2 [9,13], which is a factor $2^{(N-1)/2}$ smaller than the quantum maximum using entangled states. This bound for LHV models is sharp since the maximum is attained by choosing the LHV expectation values $\langle\sigma_x^i\rangle=\langle\sigma_y^i\rangle=1$ for all $i\in\{1,\ldots,N\}$. This shows that there are exponentially increasing gaps between the values of $|\langle M^{(N)}\rangle|$ attainable by fully separable states, fully entangled states, and LHV models. This is shown in Fig. 2.

That the maximum violation of multipartite Bell inequalities allowed by quantum mechanics grows exponentially with N with respect to the value obtainable by LHV models has been known for quite some years [28,34]. However, it is equally remarkable that the maximum value obtainable by separable quantum states *exponentially decreases* in comparison to the maximum value obtainable by LHV models, cf. Fig. 2. We thus see exponential divergence between separable quantum states and LHV theories: as N grows, the latter are able to give correlations that need more and more entanglement in order to be reproducible in quantum mechanics.

But why does quantum mechanics have correlations larger than those obtainable by a LHV model? Here we give an argument showing that it is not the degree of entanglement but the degree of inseparability that is responsible. The degree of entanglement of a state may be quantified by the value m that indicates the m-partite entanglement of the state, and the degree of inseparability by the value of k that indicates the k-separability of the state. Now suppose we have 100 qubits. For partial separability of $k\geq 51$ no state of these 100 qubits can violate the Mermin inequality (8) above the LHV bound, although the state could be up to 50-partite entangled ($m\leq 50$). However, for $k=2$, a state is possible

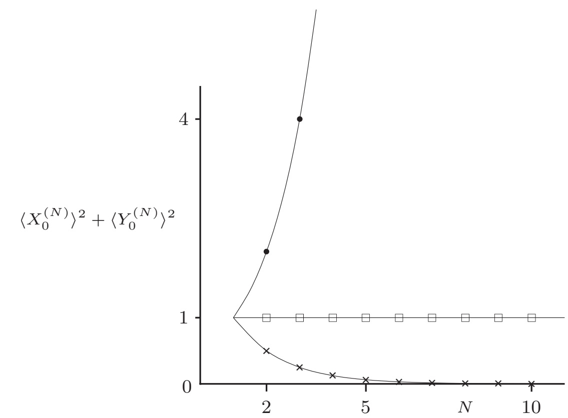

FIG. 2. The maximum value for $\langle X_0\rangle^2+\langle Y_0\rangle^2$ obtainable by entangled quantum states (dots), by separable quantum states (crosses), and by LHV models (squares), plotted as a function of the number of qubits N. Note the exponential divergence between both the maxima obtained for entangled states as well as for separable states compared to the LHV value, where the former maximum is exponentially increasing and the latter maximum is exponentially decreasing.

that is also 50-partite entangled, but which violates the Mermin inequality by an exponentially large factor of $2^{97/2}$. For $k<N$, a k-separable state is always entangled in some way, so we see that it is the degree of partial separability, not the amount of entanglement in a multiqubit state that determines the possibility of a violation of the Mermin inequality. Of course, some entanglement must be present, but the inseparability aspect of the state determines the possibility of a violation. This is also reflected in the fact that for a given N it is the value of k, and not that of m, which determines the sharp upper bounds of the Mermin inequalities.

IV. EXPERIMENTAL STRENGTH OF THE CONDITIONS FOR K-SEPARABLE ENTANGLEMENT DETECTION

Violations of the above conditions for partial separability provide sufficient criteria for detecting k-separable entanglement (and m-partite entanglement with $\lceil N/k\rceil \leq m\leq N-k+1$). It has already been shown that these criteria are stronger than the Laskowski-Żukowski criterion for k-inseparability for $k=2$, N (i.e., detecting some and full entanglement), the fidelity criterion for full inseparability (i.e., full entanglement), and the Dür-Cirac criterion for inseparability under splits. In this section we will elaborate further on the experimental usefulness and strength of these entanglement criteria when focusing on specific N-qubit states. The strength of an entanglement criterion to detect a given entangled state may be assessed by determining how well it copes with two desiderata [11]: the noise robustness of the criterion for this given state should be high, and the number of local measurements settings needed for its implementation should be small.

In this section we will first take a closer look at the issue of noise robustness and at the number of required settings for implementation of the separability criteria, both in the general state-independent case and in the case of detecting target states. We then show the strength of the criteria for a variety of specific N-qubit states.

A. Noise robustness and the number of measurement settings

White noise robustness of an entanglement criterion for a given entangled state is the maximal fraction p_0 of white noise which may be admixed to this state so that the state can no longer be detected as entangled by the criterion. Thus for a given entangled state ρ, the noise robustness of a criterion is the threshold value p_0 for which the state $\rho = p\mathbb{1}/2^N + (1-p)\rho$, with $p \geq p_0$, can no longer be detected by that criterion.

So, for the criterion for detecting full entanglement (67), the white noise robustness is found by solving the threshold equation for p_0:

$$|(1-p_0)\rho_{l,\bar{l}}| = \sum_{j\neq l} \sqrt{\left[\frac{p_0}{2^N} + (1-p_0)\rho_{j,j}\right]\left[\frac{p_0}{2^N} + (1-p_0)\rho_{\bar{j},\bar{j}}\right]}. \tag{72}$$

The state is fully entangled for $p < p_0$.

For the criterion (68), for detecting some entanglement, one finds a similar threshold equation:

$$\max_l\{|(1-p_0)\rho_{l,\bar{l}}|^2\} = \min_j\left\{\left[\frac{p_0}{2^N} + (1-p_0)\rho_{j,j}\right]\left[\frac{p_0}{2^N} + (1-p_0)\rho_{\bar{j},\bar{j}}\right]\right\}. \tag{73}$$

This equation is quadratic and easily solved. Again, the state is entangled for $p < p_0$.

A local measurement setting [35–37] is an observable such as $\mathcal{M} = \sigma_1 \otimes \sigma_l \cdots \otimes \sigma_N$, where σ_l denote single qubit observables for each of the N qubits. Measuring such a setting (determining all coincidence probabilities of the 2^N outcomes) also enables one to determine the probabilities for observables like $\mathbb{1} \otimes \sigma_2 \cdots \otimes \sigma_N$, etc. [15]. Now consider the observables $X_x^{(N)}$ and $Y_x^{(N)}$ that appear in the separability criteria of Eqs. (49)–(60). As it is easily seen from their definitions in Eq. (48), one can measure such an observable using 2^N local settings. However, these same 2^N settings then suffice to measure the observables $X_x^{(N)}$ and $Y_x^{(N)}$ for all other x since these are linear combinations of the same settings. Thus 2^N measurement settings are sufficient to determine $\langle X_x^{(N)}\rangle$ and $\langle Y_x^{(N)}\rangle$ for all x. It remains to determine the number of settings needed for the terms $\langle I_x^{(N)}\rangle$ and $\langle Z_x^{(N)}\rangle$. For all x these terms contain only two single-qubit observables: $Z^{(1)}$ and $I^{(1)} = \mathbb{1}$. They can thus be measured by a single setting, i.e., $(Z^{(1)})^{\otimes N}$.

Thus in total 2^N+1 settings are needed in order to test the separability conditions. This number grows exponentially with the number of qubits. However, this is the price we pay for being so general, i.e., for having criteria that work for all states. If we apply the criteria to detecting forms of inseparability and entanglement of specific entangled N-qubit states, this number can be greatly reduced. Knowledge of the target state enables one to select a single separability inequality for an optimal value of x in Eqs. (49)–(54) and (56)–(60). Violation of this single inequality is then sufficient for detecting the entanglement in this state, and, as we will now show, the required number of settings then grows only linear in N, with $N+1$ being the optimum for many states of interest.

For simplicity, assume that the local observables featuring in the criteria are the Pauli spin observables with the same orientation for each qubit. We can then readily use the density matrix representations of the separability criteria given at the end of each section in Sec. III. Choosing the local observables differently amounts to performing suitable bases changes to the density matrix representations and would not affect the argument.

The matrix representations of the conditions show that only some antidiagonal matrix elements and the values of some diagonal matrix elements have to be determined in order to test whether these inequalities are violated. Indeed, observe that for all x $\langle I_x^{(N)}\rangle^2 - \langle Z_x^{(N)}\rangle^2 = 4\rho_{j,j}\rho_{\bar{j},\bar{j}}$ with $\bar{j} = d+1-j$ for some $j \in \{1,2,\ldots,d\}$ and $\langle X_x^{(N)}\rangle^2 - \langle Y_x^{(N)}\rangle^2 = 4|\rho_{j,\bar{j}}|^2$ denotes some antidiagonal matrix element. It suffices to consider $x=0$ since conditions for other values of x are obtained by some local unitary basis changes that will be explicitly given later on. We now want to rewrite the density matrix representation for this single separability inequality with $x=0$ in terms of less than 2^N+1 settings.

Determining the diagonal matrix elements requires only a single setting, namely $\sigma_z^{\otimes N}$. Next, we should determine the modulus of the far-off antidiagonal element $\rho_{1,d}$ ($d=2^N$) by measuring $X_0^{(N)}$ and $Y_0^{(N)}$, since $\langle X_0^{(N)}\rangle = 2\,\mathrm{Re}\,\rho_{1,d}$ and $\langle Y_0^{(N)}\rangle = 2\,\mathrm{Im}\,\rho_{1,d}$ [cf. Eq. (61)]. Following the method of [15], these matrix elements can be obtained from two settings $\mathcal{M}_l$ and $\widetilde{\mathcal{M}}_l$, given by

$$\mathcal{M}_l = \left[\cos\left(\frac{l\pi}{N}\right)\sigma_x + \sin\left(\frac{l\pi}{N}\right)\sigma_y\right]^{\otimes N}, \quad l = 1,2,\ldots,N, \tag{74}$$

$$\widetilde{\mathcal{M}}_l = \left[\cos\left(\frac{l\pi + \pi/2}{N}\right)\sigma_x + \sin\left(\frac{l\pi + \pi/2}{N}\right)\sigma_y\right]^{\otimes N}, \quad l = 1,2,\ldots,N. \tag{75}$$

These operators obey

$$\sum_{l=1}^{N} (-1)^l \mathcal{M}_l = N X_0^{(N)}, \tag{76}$$

$$\sum_{l=1}^{N} (-1)^l \widetilde{\mathcal{M}}_l = N Y_0^{(N)}. \tag{77}$$

The proof of Eq. (76) is given in [15] and Eq. (77) can be proven in the same way.

These relations show that the imaginary and the real part of an antidiagonal element can be determined by the N settings $\mathcal{M}_l$ and $\widetilde{\mathcal{M}}_l$, respectively. This implies that the biseparability condition (67) needs only $2N+1$ measurement settings. However, if each antidiagonal term is real valued (which is often the case for states of interest) it can be determined by the N settings $\mathcal{M}_l$, so that in total $N+1$ settings suffice.

Implementation of the criteria for other x involves determining the modulus of some other antidiagonal matrix element instead of the far-off antidiagonal element $\rho_{1,d}$. The settings that allow for this determination can be obtained from a local unitary rotation on the settings $\mathcal{M}_l$ and $\tilde{\mathcal{M}}_l$ needed to measure $|\rho_{1,d}|$. This can be done as follows.

Suppose we want to determine the modulus of the matrix element $\rho_{j,\bar{j}}$. The unitary rotation to be applied is given by $U_j = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_N}$ with $j = j_1 j_2 \cdots j_N$ in binary notation, with $\sigma_0 = \mathbb{1}$ and $\sigma_1 = \sigma_x$. The settings that suffice are then given by $\mathcal{M}_{j,l} = U_j \mathcal{M}_l U_j^\dagger$ and $\tilde{\mathcal{M}}_{j,l} = U_j \tilde{\mathcal{M}}_l U_j^\dagger$ $(l = 1, 2, \ldots, N)$. For example, take $N=4$ and suppose we want to determine $\rho_{5,4}$. We obtain the required settings by applying the local unitary $U_5 = \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \sigma_x$ (since the binary notation of 5 on four bits is 0101) to the two settings $\mathcal{M}_l$ and $\tilde{\mathcal{M}}_l$ given in Eqs. (74) and (77), respectively, that for $N=4$ allow for determining $|\rho_{1,16}|$. In conclusion, using the above procedure the modulus of each antidiagonal element can be determined using $2N$ settings, and in case they are real (or imaginary) N settings suffice.

Since the strongest separability inequality for the specific target state under consideration is chosen, this reduction in the number of settings does not reduce the noise robustness for detecting forms of entanglement as compared to that obtained using the entanglement criteria in terms of the usual settings $X_x^{(N)}$, etc.

In conclusion, if the state to be detected is known, the $2N$ settings of Eqs. (74) and (75) together with the single setting $\sigma_z^{\otimes N}$ suffice, and in case this state has solely real or imaginary antidiagonal matrix elements only $N+1$ settings are needed. The white noise robustness using these settings is just as great as using the general condition that use the observables $X_x^{(N)}$ and $Y_x^{(N)}$, and is found by solving Eqs. (72) or (73) for detecting full and some entanglement, respectively.

As a final note, we observe that in order to determine the modulus of not just one but of all antidiagonal matrix elements it is more efficient to use the observables $X_x^{(N)}$, $Y_x^{(N)}$ than the observables of Eqs. (74) and (75). The first method needs 2^N settings to do this and the second needs $2^N N/2$ settings (since there are $2^N/2$ independent antidiagonal elements), i.e., the latter needs more settings than the former for all N.

Let us apply the above procedure to an example, taken from Ref. [15], the so-called four-qubit singlet state, which is given by

$$|\Phi_4\rangle = \frac{1}{\sqrt{3}}\left(|0011\rangle + |1100\rangle - \frac{1}{2}(|01\rangle + |10\rangle) \otimes (|01\rangle + |10\rangle)\right). \quad (78)$$

For detecting it as fully entangled Eq. (72) gives a noise robustness $p_0 = 12/29 \approx 0.41$, and for detecting it as entangled Eq. (73) gives a noise robustness of $16/19 \approx 0.84$. The implementation needs $16+1=17$ settings.

This number of settings can be reduced by using the fact that this state has only real antidiagonal matrix elements and that we need only look at the largest antidiagonal element. As shown above, this matrix element can be measured in four settings. Thus the total number of settings required is reduced to only five. The off-diagonal matrix element to be determined is $|0011\rangle\langle 1100|$. The four settings that allow for this determination are obtained from the four settings given in Eq. (74) by applying the unitary operator $U_3 = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x \otimes \sigma_x$ to these settings.

For comparison, note that in Ref. [15] it was shown that the so-called projector-based witness for the state (78) detects full entanglement with a white noise robustness $p_0 = 0.267$ and uses 15 settings, whereas the optimal witness from [15] uses only three settings and has $p_0 = 0.317$. Here we obtain $p_0 \approx 0.41$ using five settings, implying a significant increase in white noise robustness using only two settings more.

This example gives the largest noise robustness when the conditions are measured in the standard z basis. However, sometimes one obtains larger noise robustness when the state is first rotated so as to be expressed in a different basis before it is analyzed. For example, consider the four qubit Dicke state $|2,4\rangle$, where $|l,N\rangle = \binom{N}{l}^{-1/2} \sum_k \pi_k(|1_1, \ldots, 1_l, 0_{l+1}, \ldots, 0_N\rangle)$ are the symmetric Dicke states [38] [with $\{\pi_k(\cdot)\}$ the set of all distinct permutations of the N qubits]. In the standard basis this state does not violate any of the separability conditions we have discussed above. However, if each qubit is rotated around the x axis by 90° all of the separability conditions can be violated with quite high noise robustness. Indeed, it is detected as inseparable under all splits through violation of conditions (50) for $p < p_0 = 16/19 \approx 0.84$ and as fully entangled through violation of condition (52) for $p < p_0 = 4/11 \approx 0.36$ using five settings. For comparison, Chen *et al.* [16] used specially constructed entanglement witnesses for detection of full entanglement in these states, and they obtained as noise robustness $p_0 = 2/9 \approx 0.22$ using only two settings. We have not performed an optimization procedure, so it is unclear whether or not the values obtained for p_0 can be improved.

B. Noise and decoherence robustness for the N-qubit GHZ state

In this section we determine the robustness of our separability criteria for detecting the N-qubit GHZ state in five kinds of noise processes (admixing white and colored noise, and three types of decoherence: depolarization, dephasing, and dissipation of single qubits). We give the noise robustness as a function of N for detecting some entanglement, inseparability with respect to all splits and full entanglement. We compare the results for white noise robustness of the criteria for full entanglement to that of the fidelity criterion (10) and to that of the so-called stabilizer criteria of Refs. [11,39].

The N-qubit GHZ state $|\Psi_{\mathrm{GHZ},0}^N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ can be transformed into a mixed state ρ_N by admixing noise to this state or by decoherence. Let us consider the following five such processes.

(i) Mixing in a fraction p of white noise (also called "generalized Werner states" [40]) gives

$$\rho_N^{(\mathrm{i})} = (1-p)|\Psi_{\mathrm{GHZ},0}^N\rangle\langle\Psi_{\mathrm{GHZ},0}^N| + p\frac{\mathbb{1}}{2^N}. \quad (79)$$

(ii) Mixing in a fraction p of colored noise [17] gives

$$\rho_N^{(\mathrm{ii})} = (1-p)|\Psi^N_{\mathrm{GHZ},0}\rangle\langle\Psi^N_{\mathrm{GHZ},0}| + \frac{p}{2}(|0\cdots0\rangle\langle0\cdots0| + |1\cdots1\rangle\langle1\cdots1|). \quad (80)$$

(iii) A depolarization process [18] with a depolarization degree p of a single qubit gives

$$\rho_N^{(\mathrm{iii})} = \frac{1}{2}\left\{\left[\left(1-\frac{p}{2}\right)|0\rangle\langle0| + \frac{p}{2}|1\rangle\langle1|\right]^{\otimes N} + \left[\frac{p}{2}|0\rangle\langle0| + \left(1-\frac{p}{2}\right)|1\rangle\langle1|\right]^{\otimes N} + (1-p)^N(|0\rangle\langle1|^{\otimes N} + |1\rangle\langle0|^{\otimes N})\right\}. \quad (81)$$

(iv) A dephasing process [18] with a dephasing degree p of a single qubit gives

$$\rho_N^{(\mathrm{iv})} = \frac{1}{2}[|0\rangle\langle0|^{\otimes N} + |1\rangle\langle1|^{\otimes N} + (1-p)^N(|0\rangle\langle1|^{\otimes N} + |1\rangle\langle0|^{\otimes N})]. \quad (82)$$

(v) A dissipation process [18] with a dissipation degree p of a single qubit (where the ground state is taken to be $|0\rangle$) gives

$$\rho_N^{(\mathrm{v})} = \frac{1}{2}\{|0\rangle\langle0|^{\otimes N} + [p|0\rangle\langle0| + (1-p)|1\rangle\langle1|]^{\otimes N} + (1-p)^{N/2}(|0\rangle\langle1|^{\otimes N} + |1\rangle\langle0|^{\otimes N})\}. \quad (83)$$

We now consider the question for what values of p these states $\rho_N^{(\mathrm{i})}$ to $\rho_N^{(\mathrm{v})}$ are detected as (i) containing some entanglement by the condition (58), and (ii) inseparable under any split by the conditions of the form (50) for all bipartite splits. In other words, we determine the noise (or decoherence) robustness of violations of all these conditions for $\rho_N^{(\mathrm{i})}$ to $\rho_N^{(\mathrm{v})}$. We find the following threshold values p_0:

$$\text{(i)} \quad p_0 = \frac{1}{1+2^{(1-N)}},$$

$$\text{(ii)} \quad p_0 = 1, \quad \forall N,$$

$$\text{(iii)} \quad (1-p_0)^N = \left(1-\frac{p_0}{2}\right)^{\alpha}\left(\frac{p_0}{2}\right)^{(N-\alpha)} + \left(1-\frac{p_0}{2}\right)^{(N-\alpha)}\left(\frac{p_0}{2}\right)^{\alpha},$$

$$\text{(iv)} \quad p_0 = 1, \quad \forall N,$$

$$\text{(v)} \quad p_0 = 1, \quad \forall N, \quad (84)$$

For cases (i), (ii), (iv), and (v) the threshold values p_0 for detecting some entanglement and inseparability with respect to all splits are the same because for these cases the product of the diagonal matrix elements $\rho_{j,j}\rho_{\bar{j}\bar{j}}$ is the same for all $j\neq 1,d$. Only in case (iii) is this product different for different j. We then have to take the minumum and maximum value, respectively, from which it follows that α is to be set to $[N/2]$ for detecting some entanglement and to 1 for detecting inseparability with respect to all splits. Here $[N/2]$ is the largest integer smaller or equal to $N/2$.

The result in case (i) is in accordance with the results of Refs. [2,4], where it is furthermore shown that the opposite holds as well, i.e., if and only if $p<1/(1+2^{(1-N)})$ then $\rho_N^{(\mathrm{i})}$ is inseparable under any split and otherwise it is fully separable. Thus all states of the form (79) that are inseparable under any split are detected by violations of the conditions of the form (50) for all bipartite splits. The same holds for cases (ii), (iv), and (v), since all states $\rho_N^{(\mathrm{ii})}$, $\rho_N^{(\mathrm{iv})}$, and $\rho_N^{(\mathrm{v})}$ are inseparable under any split for all $p<1$. In other words, as soon as a fraction of the GHZ state is present, these states are inseparable under any split. In case (i) p_0 increases monotonically from $p_0=2/3$ for $N=2$ to $p_0=1$ for large N. For process (iii) these limiting values are not so straightforward: $p_0=(3-\sqrt{3})/3\approx0.42$ for $N=2$, and $p_0=(5-\sqrt{5})/5\approx0.55$ for large N. In conclusion, the noise and decoherence robustness is high for all N, except maybe for case (iii).

Next, consider the noise robustness for detecting full entanglement by means of the biseparability condition (52). The result is the following:

$$\text{(i)} \quad p_0 = 1/[2(1-2^{-N})],$$

$$\text{(ii)} \quad p_0 = 1, \quad \forall N,$$

$$\text{(iii)} \quad p_0 \approx 0.42, 0.28, 0.22, 0.18, \quad N=2,3,4,5,$$

$$\text{(iv)} \quad p_0 = 1, \quad \forall N,$$

$$\text{(v)} \quad p_0 \approx 1, 0.48, 0.39, 0.35, \quad N=2,3,4,5. \quad (85)$$

For case (i) the noise robustness is equivalent to the fidelity criterion (10). For large N p_0 decreases to the limit value $p_0=1/2$. Cases (ii) and (iv) have $p_0=1$, thus as soon as the states $\rho_N^{(\mathrm{ii})}$ and $\rho_N^{(\mathrm{iv})}$ are entangled they are fully entangled. For cases (iii) and (v) we listed the noise robustness found numerically for $N=2$ to 5. These values decrease for increasing N.

Let us compare the results for white noise robustness [case (i)] to the results obtained from the so-called stabilizer formalism. This formalism [41] is used by Tóth and Gühne to derive entanglement witnesses [11,39] that are especially useful for minimizing the number of settings required to detect either full or some entanglement. Here we will only consider the criteria formulated for detecting entanglement of the N-qubit GHZ states. The stabilizer witness by Tóth and Gühne that detects some entanglement has $p_0=2/3$, independent of N, and requires only three settings [cf. Eq. (13) in [11]]. The strongest witness for full entanglement of Tóth and Gühne has a robustness $p_0=1/(3-2^{(2-N)})$ and requires only two settings [cf. Eq. (23) in [11]].

Figure 3 shows these threshold noise ratios for detecting full entanglement for these three criteria. Note that the criterion of Tóth and Gühne [11] needs only two measurement settings, whereas our criteria need $N+1$ settings. So although the former are less robust against white noise admixture, they compare favorably with respect to minimizing the number of measurement settings.

Although we give a criterion for full entanglement that is generally stronger than the fidelity criterion, for the N-partite GHZ state this does not lead to better noise robustness. It appears that for large N the noise threshold $p_0=1/2$ is the best one can do. However, in the limit of large N the GHZ state is inseparable under all splits for all $p_0<1$, as was shown in (i) in Eq. (84), see also Fig. 3. Furthermore, we have seen that if the state $\rho_N^{(i)}$ (i.e., the GHZ state with a fraction p of white noise) is entangled it is also inseparable under any split. Because of the high symmetry of both the GHZ state and white noise, one might conjecture that if the state $\rho_N^{(i)}$ is entangled it is also fully entangled. At present, however, it is unknown whether this is indeed true. Detecting the states $\rho_N^{(i)}$ as fully entangled appears to be a much more demanding task than detecting them as inseparable under all splits. In the first case, for large N, only a fraction of 50% noise is permitted, in the second case one can permit any noise fraction (less than 100%). Note that we have given explicit examples of states that are diagonal in GHZ basis [cf. Eq. (14) of Sec. II B], and that are inseparable under any split, but not fully entangled; but these are not of the form $\rho_N^{(i)}$.

Lastly, we mention that our criteria detect the various forms of entanglement and inseparability also if the state $|\Psi^N_{\text{GHZ},0}\rangle$ is replaced by any other maximally entangled state [i.e., any state of the GHZ basis, cf. Eq. (13)], a feature which is not possible using linear entanglement witnesses. There is no single linear witness that detects entanglement of all maximally entangled states.

C. Detecting bound entanglement for $N\geqslant 3$

Violation of the separability inequality (58) allows for detecting all bound entangled states of Ref. [42]. These states have the form

$$\rho_B=\frac{1}{N+1}\left(|\Psi^N_{\text{GHZ},\alpha}\rangle\langle\Psi^N_{\text{GHZ},\alpha}|+\frac{1}{2}\sum_{l=1}^{N}P_l+\bar{P}_l\right),\quad(86)$$

with P_l the projector on the state $|0\rangle_1\cdots|1\rangle_l\cdots|0\rangle_N$, and where $\bar{P}_l$ is obtained from P_l by replacing all zeros by ones and vice versa. For $N\geqslant 4$ these states are entangled and have positive partial transposition (PPT) with respect to transposition of any qubit. This means they are bound entangled [43]. Note that they are detected as entangled by the N-partite Mermin inequality $|M_N|\leqslant 2$ of Sec. III C only for $N\geqslant 8$ [42]. However, the condition (58) detects them as entangled for $N\geqslant 4$. Thus all bound entangled states of this form are detected as entangled by this latter condition. The white noise robustness for this purpose is $p_0=2^N/(2+2N+2^N)$, which for $N=4$ gives $p_0=8/13\approx 0.615$ and goes to 1 for large N. Note that for $N=4$, this state violates the condition for 4-separability, and the condition for 3-separability (60), but not the condition for 2-separability. It is thus at least 2-separable entangled. It is not detected as fully entangled by these criteria. (Of course, it could still be fully entangled since these criteria are only sufficient and not necessary for entanglement.) For general N we have not investigated the k-separable entanglement of the states (86), although this can be readily performed using the criteria of Eq. (60).

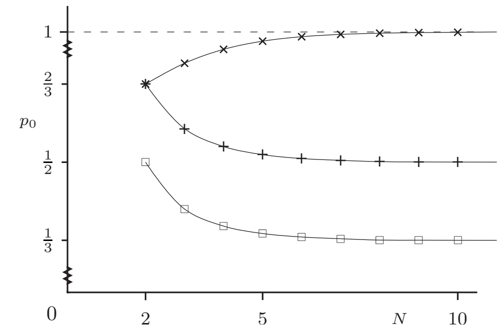

FIG. 3. The threshold noise ratios p_0 for detection of full N-qubit entanglement when admixing white noise to the N-qubit GHZ state for the criterion (52) derived here (plus signs) and for the stabilizer witness of Ref. [11] (squares). The noise robustness for detecting inseparability under all splits as given in (i) in Eq. (84) is also plotted (crosses).

Another interesting bound entangled state is the so-called four-qubit Smolin state [44]

$$\rho_S=\frac{1}{4}\sum_{j=1}^{4}|\Psi^j_{ab}\rangle\langle\Psi^j_{ab}|\otimes|\Psi^j_{cd}\rangle\langle\Psi^j_{cd}|,\quad(87)$$

where $\{|\Psi^j\rangle\}$ is the set of four Bell states $\{|\phi^\pm\rangle,|\psi^\pm\rangle\}$, and a,b,c,d label the four qubits. This state is also detected as entangled by the criterion (58), and with white noise robustness $p_0=2/3$. The Smolin state violates the separability conditions (50) for biseparability under the splits a-(bcd), b-(acd), c-(abd), d-(abc). However, it is separable under the splits (ab)-(cd), (ac)-(bd), (ad)-(bc) (cf. [44]). This state is thus inseparable under splits that partition the system into two subsets with one and three qubits, but it is separable when each subset contains two qubits.

So far we have detected bound entanglement for $N\geqslant 4$. What about $N=3$? Consider the three-qubit bound entangled state of [3]:

$$\rho=\frac{1}{3}|\Psi^3_{\text{GHZ},0}\rangle\langle\Psi^3_{\text{GHZ},0}|+\frac{1}{6}(|001\rangle\langle 001|+|010\rangle\langle 010|+|101\rangle\langle 101|+|110\rangle\langle 110|).\quad(88)$$

This state is detected as entangled by the criterion (35), with white noise robustness $p_0=4/7\approx 0.57$. It violates the biseparability condition (28) for the split a-(bc) so it is at least biseparable entangled, but does not violate the condition (34) for biseparability, i.e., it is not detected as fully entangled. In fact, it can be shown using the results of Ref. [4] that this state is separable under the splits b-(ac) and c-(ab).

V. DISCUSSION

We have discussed partial separability of quantum states by distinguishing k-separability and α_k-separability and used these distinctions to extend the classification proposed by

Dür and Cirac. We discussed the relationship of partial separability to multipartite entanglement and distinguished the notions of a k-separable entangled state and a m-partite entangled state and indicated the interrelations of these kinds of entanglement.

Next, we have presented necessary conditions for partial separability in the hierarchic separability classification. These are formulated in terms of experimentally accessible correlation inequalities for operators defined by products of local orthogonal observables. Violations of these inequalities provide, for all N-qubit states, criteria for the entire hierarchy of k-separable entanglement, ranging from the levels $k=1$ (full or genuine N-particle entanglement) to $k=N$ (full separability, no entanglement), as well as for specific classes within each level. Choosing the Pauli matrices as the locally orthogonal observables provided matrix representations of the criteria that bound antidiagonal matrix elements in terms of diagonal ones.

Further, the N-qubit Mermin-type separability inequalities for partial separability were shown to follow from the partial separability conditions derived in this paper. The biseparability conditions are stronger than the fidelity criterion and the Laskowski-Żukowski criterion, and the latter criterion is also shown to be strengthened for full separability and biseparability. For separability under splits the conditions are stronger than the Dür-Cirac conditions. Violation of these conditions thus give entanglement criteria that detect more entangled states than violations of these three other separability conditions.

We have furthermore shown that the required number of measurement settings for implementation of these criteria, which is 2^N+1 in general, can be drastically reduced if entanglement of a given target state is to be detected. In that case, it may be reduced to $2N+1$, and for multiqubit states with either real or imaginary antidiagonal matrix elements, only $N+1$ settings are needed.

When comparing the entanglement criteria to other state-specific multiqubit entanglement criteria it was found that the white noise robustness was high for a great variety of interesting multiqubit states, whereas the number of required settings was only $N+1$. However, these other state-specific entanglement criteria need less settings although for the states analyzed here they give lower noise robustness. Analyzing some specific target states shows that the entanglement criteria detect bound entanglement for $N \geqslant 3$.

Furthermore, we applied the entanglement criteria for some and full entanglement to the N-qubit GHZ state subjected to two different kinds of noise and three different kinds of decoherence. The robustness against colored noise and against dephasing turns out to be maximal (i.e., $p_0=1$) both for detecting some and full entanglement. It is remarkable that for large N the GHZ state allows for maximal white noise robustness for the state to remain inseparable under all possible splits, whereas for detecting full entanglement the best known result—to our best knowledge—only allows for a white noise robustness of $p_0=1/2$. It would be very interesting to search for full entanglement criteria that can close this gap, or if this is shown to be impossible to understand why this is the case.

Orthogonality of the local observables is crucial in the above derivation of separability conditions. It is due to this assumption that the multiqubit operators form representations of the generalized Pauli group. It would be interesting to analyze the role of orthogonality in deriving the inequalities. For two qubits it has been shown [45] that when orthogonality is relaxed the separability conditions become less strong, and we conjecture the same holds for their multiqubit analogs. Relaxing the requirement of orthogonality has the advantage that some uncertainty in the angles may be accommodated, which is desirable since in real experiments it may be hard to measure perfectly orthogonal observables.

It is also interesting that the separability inequalities are equivalent to bounds on antidiagonal matrix elements in terms of products of diagonal ones. We thus gain a novel perspective on why they allow for entanglement detection: they probe the values of antidiagonal matrix elements, which encode entanglement information about the state; and if these elements are large enough, this entanglement is detected. Note, furthermore, that compared to the Mermin-type separability inequalities we need not do much more to obtain our stronger inequalities. We must solely determine some diagonal matrix elements, and this can be easily performed using the single extra setting $\sigma_z^{\otimes N}$. It is also noteworthy that the comparison to the Mermin-type separability inequalities shows that the strength of the correlations allowed for by separable states is exponentially decreasing when compared to the strength of the correlations allowed for by LHV models.

Our recursive definition of the multipartite correlation operators [see Eq. (48)] is by no means unique. One can generate many new inequalities by choosing the locally orthogonal observables differently, e.g., by permuting their order in each triple of local observables. It could well be that combining such inequalities with those presented here yield even stronger separability conditions, as is indeed the case for pure two-qubit states, cf. [30]. Unfortunately, we have no conclusive answers for this open question.

We end by suggesting three further lines of future research. First, it would be interesting to apply the entanglement criteria to an even larger variety of N-qubit states than analyzed here, including, for example, all N-qubit graph and Dicke states. Second, the generalization from qubits to qudits (i.e., d-dimensional quantum systems) would, if indeed possible, prove very useful since strong partial separability criteria for N qudits have—to our knowledge—not yet been obtained. Finally, it would be beneficial to have optimization procedures for choosing the set of local orthogonal observables featuring in the entanglement criteria that gives the highest noise robustness for a given set of states. We believe we have chosen such optimal sets for the variety of states analyzed here, but since no rigorous optimization was performed, our choices could perhaps be improved.

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