Monogamy, entanglement and deep hidden variables

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Prospects & Introduction

I will reconsider the well-known (local) hidden variable program and the famous CHSH inequality.

Some **elementary** investigations and results are presented (by me and others) that I believe to have general repercussions.

These are intended to deepen our understanding of what it takes to violate the Bell inequality and how this relates to quantum and no-signalling correlations.

As part of the recent ‘paradigm change’ to study quantum mechanics (QM) ‘from the outside, not just from the inside’.

Study QM ‘from the outside, not just from the inside’

Motto: In order to understand quantum mechanics it is useful to demarcate those phenomena that are essentially quantum, from those that are more generically non-classical.

Investigate theories that are neither classical nor quantum; explore the space of possible theories from a larger theoretical point of view.

> *“Is quantum mechanics an island in theory space?”* (Aaronson, 2004). If indeed so, where is it?

- It is found that many non-classical properties of QM are generic within the larger family of physical theories.

Thus rather than regard quantum theory special for having the generic quantum properties, a better attitude may be to regard classical theories as special for not having them.

Methodological morale for this talk:

> *Now it is precisely in cleaning up intuitive ideas for mathematics that one is likely to throw out the baby with the bathwater.*
>
> *J.S. Bell; 'La nouvelle cuisine', 1990.*

► So I will keep things simple, and use a minimal of mathematics.

But the well-known simplest case, the setup of the EPR-Bohm experiments – already studied for over 40 years –, is not so simple after all: there is still very much to be discovered.

Outline

(I) Review of (local) hidden-variable models for the EPR-Bohm setup

(II) CHSH inequality revisited

- Strengthening Tsirelson's bound
- non-commutativity: both more and less than the classical

(III) Beyond LHV and QM

- Surface vs. subsurface level
- From non-locality to no-signalling

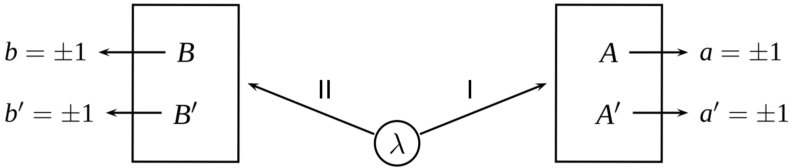
(VI) Monogamy of non-local correlations

- Review of monogamy and shareability of correlations
- Interpreting Bell's theorem

(V) Conclusion and discussion

Section I: Local realism and hidden variables

Setup of the (2,2,2) *Gedankenexperiment*:



1. One assumes that the particle pair and other relevant degrees of freedom are captured in some physical state $\lambda \in \Lambda$ ('beables').

2. The model gives the probability for obtaining outcomes a, b when measuring A, B on a system in the state λ: $P(a, b|A, B, \lambda)$.

3. Empirically accessible probabilities of outcomes are obtained by averaging over some probability density on λ:

$$P(a, b|A, B) = \int_\Lambda P(a, b|A, B, \lambda)\rho(\lambda|A, B)d\lambda.$$

Conditions imposed on the model

– Factorisability (Bell called this 'local causality'):

$$P(a, b|A, B, \lambda) = P(a|A, \lambda)P(b|B, \lambda).$$

– Independence of the Source (IS): $\rho(\lambda|A, B) = \rho(\lambda)$.

- **Consequences of the assumptions:**

Factorisability $\wedge$ IS $\implies P(a, b|A, B) = \int_\Lambda P(a|A, \lambda)P(b|B, \lambda)\rho(\lambda)d\lambda$

(all correlations are local correlations)

$\implies$ CHSH inequality is obeyed.

$$|\langle AB\rangle_{\text{lhv}} + \langle AB'\rangle_{\text{lhv}} + \langle A'B\rangle_{\text{lhv}} - \langle A'B'\rangle_{\text{lhv}}| \leq 2$$

Consider the **CHSH polynomials**: $\mathcal{B}$ and $\mathcal{B}'$, where

$$\mathcal{B} = AB + AB' + A'B - A'B'$$
$$\mathcal{B}' = A'B' + A'B + AB' - AB$$

- Then, all quantum states must obey (Uffink [2002]):

$$\max_{A,A',B,B'} \langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 8, \quad \forall\rho \in \mathcal{Q},$$

This implies (and strengthens) the Tsirelson inequality:

$$\max_{A,A',B,B'} |\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2\sqrt{2}, \quad \forall\rho \in \mathcal{Q}.$$

Separable states must obey the well-known more stringent bound:

$$\max_{A,A',B,B'} |\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2, \quad \forall\rho \in \mathcal{Q}_{\text{sep}}.$$

Orthogonal Measurements

But the maximum bound $2\sqrt{2}$ is **only** obtainable using entangled states and choosing locally *orthogonal measurements*. For qubits the latter are anti-commuting; $\{A, A'\} = 0$, $\{B, B'\} = 0$.

- Now, for any spin-$\frac{1}{2}$ state ρ on $\mathcal{H} = \mathbb{C}^2$, and any orthogonal triple of spin components A, A' and A'' ($A \perp A' \perp A''$), one has

$$\langle A \rangle_\rho^2 + \langle A' \rangle_\rho^2 + \langle A'' \rangle_\rho^2 \leq 1. \qquad (2)$$

$\Longrightarrow$ But then separable states must obey a sharper quadratic inequality:

$$\max_{A \perp A', B \perp B'} \langle \mathcal{B} \rangle_\rho^2 + \langle \mathcal{B}' \rangle_\rho^2 \leq 2, \quad \forall \rho \in \mathcal{Q}_{\text{sep}},$$

which in turn gives the linear inequalities:

$$\max_{A \perp A', B \perp B'} |\langle \mathcal{B} \rangle_\rho|, |\langle \mathcal{B}' \rangle_\rho| \leq \sqrt{2}, \quad \forall \rho \in \mathcal{Q}_{\text{sep}}.$$

- A factor $\sqrt{2}$ stronger than the original CHSH bound of 2.

On the CHSH inequality

$$\mathcal{B} = AB + AB' + A'B - A'B' \quad , \quad \mathcal{B}' = A'B' + A'B + AB' - AB.$$

$$\langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 8, \qquad \rho \in \mathcal{Q}$$

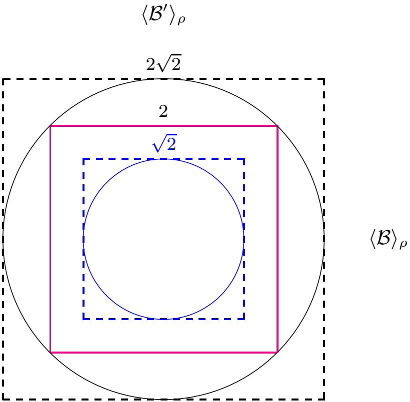
$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2\sqrt{2}, \quad \rho \in \mathcal{Q}$$

$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq 2, \qquad \rho \in \mathcal{Q}_{\text{sep}}$$

For $A \perp A', B \perp B'$:

$$\langle\mathcal{B}\rangle_\rho^2 + \langle\mathcal{B}'\rangle_\rho^2 \leq 2, \qquad \rho \in \mathcal{Q}_{\text{sep}}$$

$$|\langle\mathcal{B}\rangle_\rho|, |\langle\mathcal{B}'\rangle_\rho| \leq \sqrt{2}, \quad \rho \in \mathcal{Q}_{\text{sep}}$$



Comparing to LHV theories

It is interesting to ask whether one can obtain a similar stronger inequality as $|\langle \mathcal{B} \rangle_\rho| \leq \sqrt{2}$ in the context of local hidden-variable theories, for which we know that $|\langle \mathcal{B} \rangle_{\rm lhv}| \leq 2$.

The assumption to be added to such an LHV theory is the requirement that for any orthogonal choice of A, A' and A'' and for every given λ we have the analog of (2) which is

$$\langle A \rangle_\lambda^2 + \langle A' \rangle_\lambda^2 + \langle A'' \rangle_\lambda^2 = 1, \quad (3)$$

where $\langle A \rangle_\lambda = \sum_{a=\pm 1} a\, P(a|A, \lambda)$, etc.

- But a requirement like (3) is *by no means* obvious for a local hidden-variable theory.

Indeed, as has often been pointed out, such a theory may employ a mathematical framework which is completely different from quantum theory.

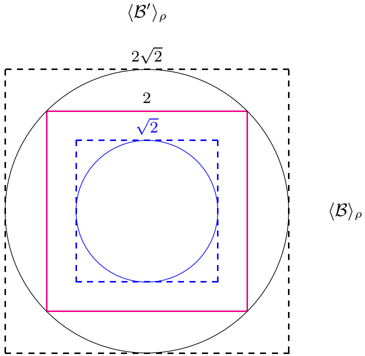
There is no *a priori* reason why the orthogonality of spin directions should have any particular significance in the hidden-variable theory, and why such a theory should confirm to quantum mechanics in reproducing $\langle A\rangle^2 + \langle A'\rangle^2 + \langle A''\rangle^2 = 1$ if one conditionalizes on a given hidden-variable state λ.

► One is reminded here of Bell's critique on von Neumann's 'no-go theorem'.

Thus, the additional requirement would appear entirely unmotivated within an LHV theory.

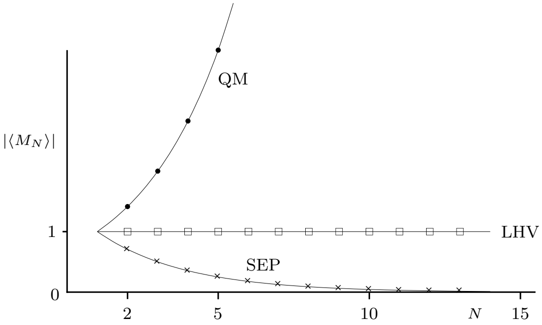
It appears that testing for entanglement within quantum theory and testing quantum mechanics against the class of all LHV theories are not equivalent issues.

- [cf. hidden nonlocality, Werner states] .



All quantum states outside the blue circle, but inside the red square, are entangled. Yet, their correlations are reproducible via a LHV model.

Multipartite generalisation



Here M_N is the Mermin polynomial (a multipartite generalisation of the CHSH polynomial $\mathcal{B}$) for orthogonal observables.

- Usually Bell inequalities are interesting only if there exist a certain quantum state that violates it. However, we here see that it is also very interesting to ask not what quantum states violate a certain Bell inequality, but what quantum states **cannot** violate such a Bell inequality, and by what factor.

- The maximum value of multipartite Bell inequalities obtainable by separable quantum states **exponentially decreases** with respect to the maximum value obtainable by LHV models. Thus as the number of particles increases a larger and larger set of LHV correlations need entanglement to be reproducible by quantum mechanics.

- It is precisely the quantum feature of **incompatible** (i.e. complementary) observables encoded via anticommutativity, which by itself is non-classical, that allows for the strange fact that a '*less than classical*' feature arises in QM.

Section III: Going beyond QM and CHSH

Surface probabilities: $P(a, b|A, B)$

Determined via measurement of relative frequencies.

Subsurface probabilities: $P(a, b|A, B, \lambda)$

Generally inaccessible, conditioned on hidden variables.

- Definitions of different kinds of bi-partite surface correlations:

a) **Local**: $P(a, b|A, B) = \int_\Lambda d\lambda\, \rho(\lambda)\, P(a|A, \lambda)P(b|B, \lambda)$.

b) **Quantum**: $P(a, b|A, B) =$ Tr$[\, M_a^A \otimes M_b^B\, \rho\,]$, $\sum_a M_a^A = \mathbb{1}$.

c) **No-signalling**: $P(a|A)^B = P(a|A)^{B'} := P(a|A)$

where $P(a|A)^B = \sum_b P(a, b|A, B)$, etc.

d) **Deterministic**: $P(a, b|A, B) \in \{0, 1\}$.

Intermezzo: Carving up factorisability

Jarrett / Shimony introduce finer distinctions that together imply factorisability.

- **Parameter Independence (PI)**:

$$P(a|A,B,\lambda) = P(a|A,\lambda) \text{ and } P(b|A,B,\lambda) = P(b|B,\lambda).$$

- **Outcome Independence (OI)**:

$$P(a,b|A,B,\lambda) = P(a|A,B,\lambda)P(b|A,B,\lambda).$$

PI $\wedge$ OI $\implies$ Factorisability: $P(a,b|A,B,\lambda) = P(a|A,\lambda)P(b|B,\lambda)$.

Surface vs. Subsurface Levels

Subsurface:

- $\text{OI} \wedge \text{PI} \implies$ Factorisability
- Determinism $\implies$ OI

(i) Deterministic hidden variables and violation of Factorisability implies violation of PI. (e.g., Bohmian mechanics)

(ii) PI and violation of Factorisability implies indeterminism at the hidden-variable level. (e.g., Leggett's recent HV model)

► **Surface analogs:** No theory can be deterministic, non-local and no-signalling. [cf. Masanes et al. (2006)]

(iii) Any deterministic non-local correlation must be signalling.

(iv) Any non-local correlation that is no-signalling must be indeterministic, i.e., the outcomes are only probabilistically predicted. (e.g., quantum mechanics, Bohm)

Determinism, yet indeterminism

Now again consider Bohmian mechanics: because it obeys no-signalling and gives rise to non-local correlations (since it violates the CHSH inequality) it **must** predict the outcomes only probabilistically.

In other words, although fundamentally (at the deeper HV level) deterministic it must necessarily be predictively indeterministic.

► Thus no 'Bohmian demon' can have perfect control over the hidden variables and still be non-local and no-signalling at the surface (as QM requires).

• This is not specific to Bohmian mechanics: **any** deterministic theory that obeys no-signalling and gives non-local correlations must have the same feature: It must predict the outcomes of measurement indeterministically. And this is *independent* of whether the theory is required to reproduce QM.

Discerning no-signalling correlations

We have seen that requiring no-signalling in conjunction with some other constraint has strong consequences.

- But what if we solely require no-signalling? Can we find a non-trivial constraint that follows from no-signalling alone?

The CHSH inequality does not suffice to discern no-signalling correlations because these can maximally violate it up to the algebraic maximum of a value of 4 (e.g., PR-boxes).

► *But an analogue does suffice*:

$$|\langle AB\rangle_{\text{ns}} + \langle A'B\rangle_{\text{ns}} + \langle A\rangle_{\text{ns}} - \langle A'\rangle_{\text{ns}}| \leq 2.$$

$\Longrightarrow$ Any correlation that violates this inequality is signalling:
$P(b|B)^A := \sum_a P(a,b|A,B) \neq \sum_a P(a,b|A',B) := P(b|B)^{A'}$.

Reproducing perfect (anti-) correlations

Suppose we want to reproduce the following perfect correlation and perfect anti-correlation using a no-signalling theory:

$$\forall\, \vec{a}, \vec{b} : \quad \langle \vec{a}\, \vec{b} \rangle = -1, \quad \text{when} \quad \vec{a} = \vec{b}$$
$$\forall\, \vec{a}, \vec{b} : \quad \langle \vec{a}\, \vec{b} \rangle = 1, \quad \text{when} \quad \vec{a} = -\vec{b}$$

(The singlet state gives such correlations, but we will not assume any quantum mechanics in what follows)

- The no-signalling inequalities give two non-trivial constraints:

$$\langle \vec{a} \rangle^{I}_{\text{ns}} + \langle \vec{a} \rangle^{II}_{\text{ns}} = 0$$

$$\langle -\vec{a} \rangle^{I}_{\text{ns}} = -\langle \vec{a} \rangle^{I}_{\text{ns}}$$

This states that the marginal expectation values for party I and II must add up to zero for measurements in the same direction, and individually they must be odd functions of the settings.

In case both systems are treated the same, i.e., $\langle\vec{a}\rangle^I_{\rm ns} = \langle\vec{a}\rangle^{II}_{\rm ns}$, the marginal expectation values must vanish: $\langle\vec{a}\rangle^I_{\rm ns} = \langle\vec{a}\rangle^{II}_{\rm ns} = 0$.

Thus all marginal probabilities must be uniformly distributed:
$P^I(+|\vec{a}) = P^I(-|\vec{a}) = 1/2$, etc.

Consequently, any no-signalling theory that reproduces perfect (anti-) correlations for all measurement directions and that treats the two systems identically, must locally have uniformly distributed marginals.

► This shows why Leggett's[2004] recent model is incabable of reproducing the singlet correlations of QM: it postulates non-flat marginals (they are to follow the Malus law).

► No appeal to QM is needed. If one does, even stronger conclusions can be derived. (Colbeck & Renner; Branciard *et al*.)

Entanglement is monogamous

If a pure quantum state of two systems is entangled, then none of the two systems can be entangled with a third system.

1. Suppose that systems a and b are in a pure entangled state.
2. Then when the system ab is considered as part of a larger system, the reduced density operator for ab must by assumption be a pure state.
3. However, for the composite system ab (or for any of its subsystems a or b) to be entangled with another system, the reduced density operator of ab must be a mixed state.
4. But since it is by assumption pure, no entanglement between ab and any other system can exist.

Monogamy because of no-cloning

This monogamy can also be understood as a consequence of the linearity of quantum mechanics that is also responsible for the no-cloning theorem.

1. For suppose that party a has a qubit which is maximally pure state entangled to both a qubit held by party b and a qubit held by party c.
2. Party a thus has a single qubit coupled to two perfect entangled quantum channels.
3. This party could exploit this to teleport two perfect copies of an unknown input state, thereby violating the no-cloning theorem, and thus the linearity of quantum mechanics.

Mixed state entanglement can be shared

The W-state $|\psi\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ has bi-partite reduced states that are all identical and entangled.

- ‘sharing of mixed state entanglement’, or ‘promiscuity of entanglement’.

But this promiscuity is not unbounded: no entangled bi-partite state can be shared with an infinite number of parties.

Here a bi-partite state ρ_{ab} is said to be N-shareable when it is possible to find a quantum state $\rho_{ab_1b_2\ldots b_N}$ such that

$$\rho_{ab} = \rho_{ab_1} = \rho_{ab_2} = \ldots = \rho_{ab_N},$$

where ρ_{ab_k} is the reduced state for parties a and b_k.

- Fannes *et al* [1988], Raggio *et al* [1989]: A bi-partite state is N-shareable for all N (also called ∞-shareable) iff it is separable.

Quantifying the monogamy of entanglement

Coffman, Kundu and Wootters [2000] gave a trade-off relation between how entangled a is with b, and how entangled a is with c in a three-qubit system abc that is in a pure state:

$$\tau(\rho_{ab}) + \tau(\rho_{ac}) \leq 4\det\rho_a$$

with $\rho_a = \mathrm{Tr}_{bc}[|\psi\rangle\langle\psi|]$ and $|\psi\rangle$ the pure three-qubit state, where $\tau(\rho_{ab})$ is the tangle between A and B, analogous for $\tau(\rho_{ac})$.

The multi-partite generalization has been recently proven by Osborne & Verstraete [2006].

Monogamy of non-local correlations

Suppose one has some no-signalling three-party probability distribution $P(a_1, a_2, a_3|A_1, A_2, A_3)$ for parties a, b and c.

- Then in case the marginal distribution $P(a_1, a_2|A_1, A_2)$ or ab is extremal (and thus non-local) it cannot be correlated to the third system c (Barrett, *et al* [2006]):

$$P(a_1, a_2, a_3|A_1, A_2, A_3) = P(a_1, a_2|A_1, A_2)P(a_3|A_3),$$

which implies that party c is completely uncorrelated with party ab: the extremal correlation $P(a_1, a_2|A_1, A_2)$ is completely monogamous.

Note that this implies that all local Bell-type inequalities for which the maximal violation consistent with no-signalling is attained by a unique correlation have monogamy constraints. An example is the CHSH inequality, as will be shown below.

Quantifying the monogamy of non-local correlations

Extremal no-signalling correlations thus show monogamy, but what about non-extremal no-signalling correlations?

► Just as was the case for quantum states where non-extremal (mixed state) entanglement can be shared, non-extremal no-signalling correlations can be shared as well.

- Toner [2006] proved a tight trade-off relation:

$$|\langle \mathcal{B}_{ab} \rangle_{\mathrm{ns}}| + |\langle \mathcal{B}_{ac} \rangle_{\mathrm{ns}}| \leq 4.$$

Extremal no-signalling correlations can attain $|\langle \mathcal{B}_{ab} \rangle_{\mathrm{ns}}| = 4$ so that necessarily $|\langle \mathcal{B}_{ac} \rangle_{\mathrm{ns}}| = 0$, and vice versa (this is monogamy of extremal no-signalling correlations), whereas non-extremal ones are shareable.

Monogamy for other kinds of correlations

$$\mathcal{B}_{ab} = AB + AB' + A'B - A'B' \quad , \quad \mathcal{B}_{ac} = AC + A'C + AC' - A'C'$$

- For general unrestricted correlations no monogamy holds, i.e., $|\langle \mathcal{B}_{ab} \rangle|$ and $|\langle \mathcal{B}_{ac} \rangle|$ are not mutually constrained.
- Quantum correlations are monogamous: $\langle \mathcal{B}_{ab} \rangle_{\mathrm{qm}}^2 + \langle \mathcal{B}_{ac} \rangle_{\mathrm{qm}}^2 \leq 8$.
- Classical correlations are not monogamous. It is possible to have both $|\langle \mathcal{B}_{ab} \rangle_{\mathrm{lhv}}| = 2$ and $|\langle \mathcal{B}_{ac} \rangle_{\mathrm{lhv}}| = 2$.
- Separable quantum state ar neither monogamous: $|\langle \mathcal{B}_{ab} \rangle_{\mathrm{qm}}|, |\langle \mathcal{B}_{ac} \rangle_{\mathrm{qm}}| \leq 2$, $\rho \in \mathcal{Q}_{\mathrm{sep}}$.

(For orthogonal measurements a stronger bound holds: $\leq \sqrt{2}$)

Monogamy of correlations

$$\mathcal{B}_{ab} = AB + AB' + A'B - A'B' \quad , \quad \mathcal{B}_{ac} = AC + A'C + AC' - A'C'$$

$|\langle\mathcal{B}_{ab}\rangle|, \ |\langle\mathcal{B}_{ac}\rangle| \leq 4$

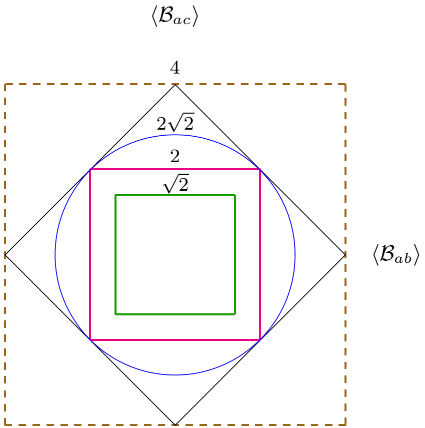
$|\langle\mathcal{B}_{ab}\rangle_{\rm ns}| + |\langle\mathcal{B}_{ac}\rangle_{\rm ns}| \leq 4$ [a]

$\langle\mathcal{B}_{ab}\rangle^2_{\rm qm} + \langle\mathcal{B}_{ac}\rangle^2_{\rm qm} \leq 8 \quad \rho \in \mathcal{Q}$ [b]

$|\langle\mathcal{B}_{ab}\rangle_{\rm lhv}|, \ |\langle\mathcal{B}_{ac}\rangle_{\rm lhv}| \leq 2$

For $A \perp A', B \perp B', C \perp C'$:

$|\langle\mathcal{B}_{ab}\rangle_{\rm qm}|, \ |\langle\mathcal{B}_{ac}\rangle_{\rm qm}| \leq \sqrt{2} \quad \rho \in \mathcal{Q}_{\rm sep}$



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[a] Toner [2006]

[b] Toner & Verstraete [2006]

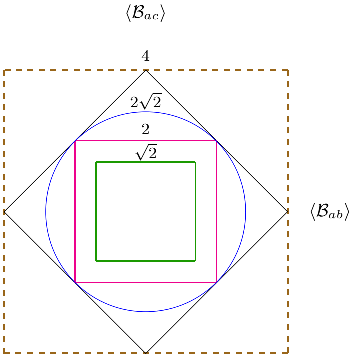
Consequences of this monogamy of correlations

In case the no-signalling correlations are non-local they can not be shared (impossible for both $|\langle\mathcal{B}_{ab}\rangle_{\mathrm{ns}}| \geq 2$ and $|\langle\mathcal{B}_{ac}\rangle_{\mathrm{ns}}| \geq 2$).

- The monogamy bound therefore gives a way of discriminating no-signalling from general correlations: if the bound is violated the correlations cannot be no-signalling (i.e., they must be signalling).
- Extremal quantum and no-signalling correlations are fully monogamous.
- Remark: The quantum bound can be strengthened to

$$\langle\mathcal{B}_{ab}\rangle_{\mathrm{qm}}^2 + \langle\mathcal{B}_{ac}\rangle_{\mathrm{qm}}^2 \leq 8(1 - \langle\sigma_y\rangle_a^2),$$

where one measures in the x-z plane.



Shareability of correlations

A **general unrestricted distribution** $P(a, b_1|A, B_1, \ldots, B_N)$ is N-shareable with respect to the second party if an $(N+1)$-partite distribution

$$P(a, b_1, \ldots, b_N|A, B_1, \ldots, B_N)$$

exists, symmetric with respect to $(b_1, B_1), (b_2, B_2), \ldots, (b_N, B_N)$ and with marginals $P(a, b_i|A, B_1, \ldots, B_N)$ equal to the original distribution $P(a, b_1|A, B_1, \ldots, B_N)$, for all i.

- If a distribution is shareable for all N it is called ∞-shareable.

Analogously: **a no-signalling distribution** $P(a, b_1|A, B_1)$ is N-shareable if the $(N+1)$-partite distribution has marginals $P(a, b_i|A, B_i)$ equal to the original distribution $P(a, b_1|A, B_1)$, for all i.

Consider a general **unrestricted** correlation $P(a, b_1|A, B_1, \ldots, B_N)$. We can then construct

$$P(a, b_1, \ldots, b_N|A, B_1, \ldots, B_N) = P(a, b_1|A, B_1, \ldots, B_N)\delta_{b_1,b_2} \cdots \delta_{b_1,b_N},$$

which has the same marginals $P(a, b_i|A, B_1, \ldots, B_N)$ equal to the original distribution $P(a, b_1|A, B_1, \ldots, B_N)$. This holds for all i, thereby proving the ∞-shareability, i.e., it can be shared for all N.

If we restrict the distributions to be **no-signalling**, Masanes, *et al* [2006] proved that ∞-shareability implies that the distribution is local, i.e., it can be written as

$$P(a, b_1, \ldots, b_N|A, B_1 \ldots, B_N) = \int_\Lambda d\lambda p(\lambda) P(a|A, \lambda) P(b_1|B_1, \lambda) \cdots P(b_N|B_N, \lambda),$$

for some local distributions $P(a|A, \lambda), P(b_1|B_1, \lambda), \ldots, P(b_N|B_N, \lambda)$ and hidden-variable distribution $p(\lambda)$.

local realism $\iff$ ∞-shareability of correlations

$\exists$ local model for $P(a,b|A,B)$ when party 1 has an arbitrary number and party 2 has N possible measurements

$\iff$

N-shareability of correlations

- Proof:

$\Longrightarrow$ classical information can be cloned indefinitely.

$\Longleftarrow$ Since $P(a,b|A,B)$ is shareable to N parties (labelled $\mathcal{B}_i$, $i = 1,\ldots,N$), the correlations between A and B_i performed on party 2 are the same as the correlations between measurements of A and B_i performed on the extra party $\mathcal{B}_i$.

Therefore, the N measurements $B_1,\ldots,B_N$ performed by party 2 can be viewed as one large measurement performed on the N parties $\mathcal{B}_i$ ($i = 1,\ldots,N$). Lastly, there always exists a local hidden variable model when one of the two parties has only one measurement.

Interpreting Bell's theorem

Bell's theorem: local realism $\implies$ CHSH inequality $\implies$ $\neg$ QM

- But we have seen:

local realism $\iff$ ∞-shareability of correlations

However, quantum correlations are not always shareable, let alone ∞-shareable.

- Schumacher [2008]: Bell's theorem is about shareability of correlations, not about local realism, since we don't need ∞-shareable to get the CHSH inequality which quantum mechanics violates.

- 2-shareability is sufficient, and this is a weaker claim than the assumption of local realism.

2-shareability implies CHSH inequality

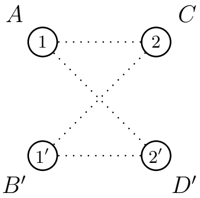
Consider an EPR-Bohm setup for part 1 and 2.

Assume that all possible correlations between 1 and 2 are shareable to another party 1' and 2' that conceivably exist. Then for the outcomes:

$$a(c+d') + b'(c-d') = \pm 2$$

which implies for the expectation values

$$|\langle AC\rangle + \langle AD'\rangle + \langle B'C\rangle - \langle B'D'\rangle| \leq 2$$



2-shareability implies

$$|\langle AC\rangle + \langle AD\rangle + \langle BC\rangle - \langle BD\rangle| \leq 2$$

The shareability justifies the counterfactual reasoning.

Section V: Conclusion and Discussion

► The simplest case is not so simple. It still gives rise to a lot of new results, when studied from a larger 'outside' point of view.

1. Non-commutativity gives 'a less than classical effect': QM generally needs entanglement to reproduce LHV correlations.

2. The conjunction of non-locality and no-signalling (as is the case in QM) is very stringent: the surface probabilities cannot be deterministic. Any determinism must stay beneath the surface.

3. Discerning no-signalling correlations from more general ones can be done via a very similar inequality as the CHSH inequality.

4. Non-local correlations (whether quantum or no-signalling) are monogamous, whereas ∞-shareability and local realism are equivalent. (A new view on Bell's theorem?)

Proof: Any deterministic no-signalling correlation must be local. [cf. Masanes et al. (2006)]

(1) Consider a deterministic probability distribution $P_{\text{det}}(a,b|AB)$.

$\Longrightarrow$ The outcomes a and b are deterministic functions of A and B: $a = a[A,B]$ and $b = b[A,B]$.

(2) Suppose it is a no-signalling distribution, then

$$P_{\text{det}}(a,b|AB) \stackrel{\text{det}}{=} \delta_{(a,b),(a[(A,B],b[A,B])} = \delta_{a,a[A,B]}\delta_{b,b[A,B]}$$
$$= P(a|A,B)P(b|A,B) \stackrel{\text{ns}}{=} P(a|A)P(b|B).$$

This is a local distribution and therefore any deterministic no-signalling correlation must be local.