Challenging the gospel: Grete Hermann on von Neumann's no-hidden-variables proof

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∞ Preliminary ∞

In 1932 **John von Neumann** had published in his celebrated book on the *Mathematische Grundlagen der Quantenmechanik*, a proof of the impossibility of theories which, by using the so-called hidden variables, attempt to give a deterministic explanation of quantum mechanical behaviors.

▶ Von Neumann's proof was sort of **holy**:

'The truth, however, happens to be that for decades nobody spoke up against von Neumann's arguments, and that his conclusions were quoted by some as the gospel.'
(F. J. Belinfante, 1973)

- In 1935 **Grete Hermann** challenged this gospel by criticizing the von Neumann proof on a fundamental point. This was however not widely known and her criticism had no impact whatsoever.

- 30 years later **John Bell** gave a similar critique, that did have great foundational impact.

∞ Outline ∞

1. **Von Neumann's 1932 no hidden variable proof**

2. **The reception of this proof**
   + John Bell's 1966 criticism

3. **Grete Hermann's critique (1935) on von Neumann's argument**

4. **Comparison to Bell's criticism**

5. **The reception of Hermann's criticism**

∞ Von Neumann's 1932 no hidden variable argument ∞

Von Neumann: What reasons can be given for the **dispersion** found in some quantum ensembles?

(Case I): The individual systems differ in **additional parameters**, which are not known to us, whose values determine precise outcomes of measurements.

$\Longrightarrow$ **deterministic hidden variables**

(Case II): 'All individual systems are in the **same state**, but the laws of nature are **not causal**'.

ad Case I:

► No **physical method** exists of dividing a dispersive ensemble into dispersion free ensembles, because of the unavoidable **measurement disturbance**.

► However, it nevertheless is possible **conceptually** to think of each ensemble as composed out of two (or more) **dispersion free** subensembles.

- Von Neumann's proof : even the latter is **impossible**.

- Von Neumann's notion of a **hidden variables** theory:

It is a causal theory which defines the state of the system 'absolutely' by supplying 'additional numerical data'– this additional data are the 'hidden parameters'.

'If we were to know all of these, then we could give the values of all physical quantities exactly and with certainty.'

A **hidden variables theory** now is one which is *' [. . .] in agreement with experiment, and which gives the statistical assertions of quantum mechanics when only* ϕ *is given (and an averaging is performed over the other coordinates).'*

▶ **Mathematical characterization** of hidden variables:

Every physically realizable state can be represented as a mixture of **homogeneous dispersion-free states**:

(α') An ensemble is **dispersion free** if

$$Exp(\mathfrak{R}^2) = (Exp(\mathfrak{R}))^2 \quad , \forall\, \mathfrak{R}.$$

(β') An ensemble is **homogeneous** or **pure** if the statistics of it is the same as that of **any** of its subensembles.

$$E = a\, E_1 + bE_2 \implies E = E_1 = E_2.$$

∞ The question of 'hidden parameters' ∞

► Now **suppose** there are homogeneous ensembles, then **if** hidden variables exist (any dispersive ensemble can thus be split into two or more non-dispersive ones), the homogeneous ensembles **must be** dispersion free:

$\Longrightarrow$ **No dispersive ensemble can be homogeneous**

This is what according to von Neumann is implied by the existence of hidden variables.

- In **classical** Kolmogorov type statistical ensembles all and only dispersion free ensembles are homogeneous.

► **What about quantum mechanical ensembles?**

Von Neumann proofs that this is not the case: **all homogeneous ensembles are dispersive**; there are no non-dispersive ensembles.

The Proof

► We have to consider a theory **general enough** to deal with both Case I and Case II statistical theories.

Von Neumann implements this as follows. Every physical **ensemble** determines a **functional** Exp, which is supposed to characterize it completely from a statistical point of view.

The Exp-functional must satisfy the following **assumptions**:

(0) To each observable of a quantum mechanical system **corresponds** a unique hypermaximal Hermitian **operator** in Hilbert space. This correspondence is **one-to-one**.

(I) If the observable $\mathfrak{R}$ has operator R then the observable $f(\mathfrak{R})$ has the operator $f(R)$.

(II) If the observables $\mathfrak{R}$, $\mathfrak{S}$, $\ldots$ have the operators R, S, $\ldots$, then the observable $\mathfrak{R} + \mathfrak{S} + \ldots$ has the operator $R + S + \ldots$. (the **simultaneous measurability** of $\mathfrak{R} + \mathfrak{S} + \ldots$ is **not** assumed.)

(A') If the observable $\mathfrak{R}$ is by nature a **nonnegative** quantity then $Exp(\mathfrak{R}) \geq 0$.

(B') If $\mathfrak{R} + \mathfrak{S} + \ldots$ are arbitrary observables and $a,\ b,\ \ldots$ real numbers, then $Exp(a\,\mathfrak{R} + b\,\mathfrak{S} + \ldots) = a\,Exp(\mathfrak{R}) + b\,Exp(\mathfrak{S}) + \ldots$.

• Von Neumann demonstrated on the basis of **all** these assumptions that there exists a **linear, semidefinite Hermitian matrix** U_{mn} such that for any observable $\mathfrak{R}$

$$Exp(\mathfrak{R}) = \sum U_{nm} R_{mn} = \mathrm{Tr}\,(\mathrm{UR}).$$

► Thus every ensemble in quantum mechanics is characterised by a **statistical operator** known as the **density operator** (or **density matrix**).

(Note: Gleason (1957) proofs this also, but requires **(B')** only for **commuting** observables, $\mathrm{Dim}(\mathcal{H}) \geq 3$.)

Von Neumann proceeds:

- What are (I) the **dispersion free** and (II) the **homogeneous ensembles** among the density operators U?

I: What U have $\mathrm{Tr}(\mathrm{UR}^2) = [\mathrm{Tr}(\mathrm{UR})]^2$ for all R?

$\Longrightarrow$ **No** U fulfill this requirement, thus **no dispersion free states exists**.

II: He next proofs that homogeneous ensembles **do exist**:

$\Longrightarrow$ The homogeneous ensembles are the **pure** quantum states (**one-dimensional** projection operators).

I & **II** $\Longrightarrow$ **All** ensembles show **dispersion**, even the homogeneous ones.

∞ Back to the question of 'hidden parameters' ∞

Can the **dispersion** in the homogeneous ensembles be explained by the fact that the states are mixtures of several states, '*which together would determine everything causally, i.e., lead to dispersion free ensembles?*'

► **Von Neumann:**

' The statistics of the homogeneous [dispersive] ensembles would then have resulted from from the averaging over all actual states of which it was composed [...]. But this is impossible for two reasons:

First, because then the homogeneous ensembles in question could be represented as a mixture of two different ensembles, contrary to its definition.

Second, because the dispersion free ensembles [. . .] do not exist.'

$\Longrightarrow$ No homogeneous ensembles exist that are dispersion free, therefore the assumption of the existence of hidden variables is refuted.

▶ John Bell paves the way for the standard view

In 1964 (published 1966) John Bell intended to show what the problem with von Neumann's argument was, after he *'saw the impossible done'*. He tracked it down to the assumption **(B')**:

$$Exp(a\,\mathfrak{R} + b\,\mathfrak{S}) = a\,Exp(\mathfrak{R}) + b\,Exp(\mathfrak{S}).$$

This relation holds true for quantum mechanics, **irrespective** of whether the operators R and S commute. (This is often not realized!)

► **Bell reasons**: It is required by von Neumann of the hypothetical dispersion free states **also**. But for a dispersion free state the expectation value must equal one of the operator's **eigenvalues**. But eigenvalues **do not** generally combine linearly.

Example: $(\sigma_x + \sigma_y)$ with eigenvalues $\pm\sqrt{2}$

$\sigma_x, \ \sigma_y$, with eigenvalues ± 1.

If applied unrestricted, the **addivity of expectation values** gives the requirement for **additivity of eigenvalues.**

► The latter is generally not true.

Bell (1966): 'The essential assumption can be criticized as follows. [. . .] A measurement of a sum of noncommuting observables cannot be made by combining trivially the results of separable observations on the two terms – it requires a quite distinct experiment. [. . .] But this explanation of the nonadditivity of allowed values also established the nontriviality of the additivity of expectation values. The latter is quite a peculiar property of quantum mechanical states, not to be expected *a priori*.'

Bell (1982): '. . . for the individual results are eigenvalues and eigenvalues of linearly related operators are not linearly related. [. . .] His very general and plausible postulate is absurd.'

► The once superior proof becomes allegedly silly

’ A few years later *Grete Hermann*, 1935, pointed out a glaring deficiency in the argument, but she seems to have been entirely ignored. Everybody continued to cite the von Neumann proof. A third of a century passed before John Bell, 1966, rediscovered the fact that von Neumann’s no-hidden-variables proof was based on an assumption that can only be described as silly [...]
(N. David Mermin, 1993)

In conclusion : the argument is unconvincing

► The additivity rule for expectation values in the case of incompatible observables cannot be justified in the light of the **Bohrian point** that contexts of measurement play a role in defining the nature of quantum reality.

'There is no reason to demand it individually of the hypothetical dispersion free states ...' (Bell, 1964). And thus there is no reason to demand that the dispersion free states are of the form of a density operator U.

- **A side remark: Bell's criticism is ironic**, it is a sort of **judo-like manouvre** (Shimony, 1984):

A Bohrian consideration **saves** hidden variables against von Neumann. '[Bell] cited Bohr in order to vindicate a family of hidden variables theories ...' (Jammer, 1974):

'[The assumption] is seen to be quite unreasonable when one remembers with Bohr "the impossibility of any sharp distinction between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which phenomena appear."' (Bell, 1964)

∞ Grete Hermann's critique on von Neumann's argument ∞

Grete Hermann published in 1935 a treatise called 'Die Naturphilosophischen Grundlagen der Quantenmechanik', published in the *Abhandlungen der Fries'schen Schule*.

Section 7: **'The circularity of von Neumann's proof'**:

$$Exp(\mathfrak{R} + \mathfrak{S}) = Exp(\mathfrak{R}) + Exp(\mathfrak{S})$$

'With this assumption the proof of von Neumann either succeeds or fails.' (*Mit dieser Voraussetzung steht und fällt der* NEUMANN*sche Beweis.*)

- Hermann first commented upon the **problematic status** of the additivity rule in the light of the impossibility of simultaneous measurement of noncommuting observables (the 'Bohrian point').

Hermann, p.100: 'Not trivial however is the relation for quantum mechanical quantities for which indeterminacy relations hold. In fact, the sum of two such quantities is not even defined [...]

Thus, for the above determined notion of a sum of two, not jointly measurable quantities, the above mentioned equation requires a proof. [...] Von Neumann concludes that for ensembles of systems with identical wave functions, and also for all ensembles, the sum rule for expectation values holds, even for such quantities that cannot be measured simultaneously.'

- This additivity rule thus needs a proof.
  Indeed, von Neumann gives one.

Von Neumann gives this proof as follows:

1) Because every ensemble of physical states can be decomposed into sub-ensembles whose elements have identical wave functions, the requirement in question has only to be proved for ensembles whose elements fulfill the requirement of having identical wave functions.

2) For these ensembles von Neumann assumes that they obey the following rule $((R+S)\phi, \phi) = (R\phi, \phi) + (S\phi, \phi)$, with ϕ the wave function of the observed system.

From this rule von Neumann concludes that for ensembles of systems with identical wave functions, and also for all ensembles, the sum rule for expectation values holds, even for such quantities that cannot be measured simultaneously.

- **Hermann proceeds**: the interpretation of the expression $(R\phi, \phi)$ is crucial for the entire proof.

"Until the proof of the impossibility of new characteristics is found, *which is still to be given here*, the expression $(R\phi, \phi)$ can only be taken to express the expectation value of the $\mathfrak{R}$-measurement of ensembles of physical systems that are *required to be* in the state ϕ";

► “It has to remain an open issue, in order for it [the assumption] to stay applicable, whether or not this expectation value is the same in all sub-ensembles that can be distinghuised by arbitrary new characteristics.”

► “If this indeed remains open, then one can no longer deduce from the sum-rule which holds for $(R\phi, \phi)$, that also in these sub-ensembles the expectation value of the sum of physical quantities equals the sum of the expectation values.”

”But with this a necessary step in Von Neumann’s proof collapses.”

"If one - just like Von Neumann - does not give up this step, then one has tacitly assumed the unproven presupposition that the elements of an ensemble of physical systems characterized by ϕ cannot have any distinguishing characteristics on which the outcome of R is dependent. *The impossibility of such a specification is just the thesis to be proved. The proof runs into a circularity.*"

► Hermann concluded, just like Bell, that von Neumann **precluded** the non-existence of dispersion free states because the additivity rule is required at the hidden variable level, without there being a justification for it.

My view — the more appropriate conclusion is:

(i) not that this precludes hidden variables but that the proof holds only for *a limited class of hidden variables*, namely those that obey **(B')**; (cf. Jeff Bub's recent point)

(ii) it is a *completeness proof* since the inclusion of the additivity postulate does not admit non-quantum mechanical ensembles. It may even be regarded a *consistency proof* of this formalism with its usual interpretaion.

Von Neumann proofs that a certain limited class of hidden variables is excluded. Those that obey his assumptions. He himself thought he was completely general and unrestrictive.

> 'Nevertheless, under all circumstances, $Exp\,(\mathfrak{R} + \mathfrak{S}) = Exp\,(\mathfrak{R}) + Exp\,(\mathfrak{S})$.' (von Neumann)

- Why? Because it holds true in quantum mechanics:

'In each state ϕ the expectation values behave additively: $(R\phi, \phi) + (S\phi, \phi) = ((R+S)\phi, \phi)$. The same holds for several summands. We now incorporate this fact into our general set-up (at this point not yet specialized to quantum mechanics).'

2) The additivity rule, although it has no justification, nevertheless does hold **true** in quantum mechanics. It is thus **surprising** that it holds true.

A priori no statistical results can be expected between $Exp(\mathfrak{R})$ and $Exp(\mathfrak{S})$. The additivity holds because '**it so happens** that the other axioms and postulates of quantum theory conspire to make $Exp(\mathfrak{R})$ expressible at $\int \psi^*\, R\, \psi\, dx$.' (Belinfante, 1973).

► Comparison to Bell's 1966 paper

(1) Bell and Hermann use 'the Bohrian point' of incompatible observables to argue against the additivity rule. It cannot be reasonably assumed at the hidden variable level, but von Neumann does so.

- Hermann: the proof therefore is implicitly circular.
- Bell: this incorrectly implies additivity of eigenvalues, and he gives an explicit counterexample.

Continued: Comparison to Bell's 1966 paper

(2) Bell furthermore addresses other no-go theorems (e.g. Jauch-Piron) and discusses the important theorem by Gleason, and produces and independent Kochen-Specker like proof *avant la lettre*.

(3) Bell addresses Bohm's hidden variable theory ('I saw the impossible done'), and indicates nonlocality.

- What did Hermann use her criticism for?

Not to argue in favour of hidden variables!

Grete Hermann shows a flaw in this proof, and concludes that notwithstanding the strength of the mathematical formalism, it cannot be deduced that further undiscovered relations with a different mathematical formulation are not possible.

G. Paparo (2012): "The mathematical formalism alone is not able to answer the question of whether the limits in the predictability of quantum mechanics are insurmountable or only there due to our lack of knowledge

With this conviction, she proceeds to show how mathematical and statistical arguments have failed to defend the causality principle and consequently, that only philosophy can answer the question of whether it is possible to overcome the limits in the predictability of quantum mechanics. "

- Further suggestion: Hermann could not / dared not be revolutionary? She stressed continuity – not radical changes.

∞ The reception of her work ∞

With hindsight we can say that Grete Hermann was **ahead of her time**. Only really after John Bell (1966) the limited applicability of von Neumann's proof becomes known.

► von Weizsäcker surely knew of her criticism (and what about Heisenberg?).

Why was her criticism ignored at the time?

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1. Von Neumann's proof was sort of **holy**:

'The truth, however, happens to be that for decades nobody spoke up against von Neumann's arguments, and that his conclusions were quoted by some as the gospel.' (F. J. Belinfante, 1973)

'Now the mere mention of concealed variables is sufficient to automatically elicit from the elect the remark that John von Neumann gave absolute proof that this way out is not possible. To me it is a curious spectacle to see the unanimity with wich the members of a certain circle accept the rigor of von Neumann's proof' (Bridgman, 1960)

‘He [Bohr] came for a public lecture.... At the end of the lecture he left and the discussion proceeded without him. Some speakers attacked his qualitative arguments –there seems lots of loopholes. The Bohrians did not clarify the arguments; they mentioned the alleged proof by von Neumann and that settled the matter.[...]. Yet, like magic, the mere name of ”von Neumann” and the mere word ”proof” silenced the objectors.’ (Feyerabend)

Why was her criticism ignored at the time?

2. Grete Herman published her treatise in a not well known series of books. The summary that appeared in 1935 in the good and well-read journal 'Die Naturwissenschaften' did not contain the argument against von Neumann, but only her Kantian ideas.

3. Bohr and Heisenberg had some interest in preserving the belief in the results of von Neumanns no hidden variables proof, since it supported their ideas.

- Sociological reasons?

1. She was a woman, in a time in which women were still not well received in the scientific community

2. She was young and without influential connections

3. She came from a different background: she had not studied physics, but philosophy and mathematics

4. She was a political outsider and dissenter

- G. Paparo (2012): Hermann has a different agenda herself:

“ I claim that one of the most relevant reasons has been overlooked in previous discussions on the matter. In my opinion, a primary reason is to be found in Hermanns personality and work, as she did not actively pursue the wider dissemination and understanding of her discovery.

First of all, she published the critical analysis of von Neumanns proof written in small font, having earlier stated that anything in such a font could be readily skipped.

Secondly, in later editions of the paper, the disproof is simply left out, stressing again the insignificance Hermann ascribed to it.

► Either way, the result of her critique to von Neumanns proof only served to show how it was impossible to answer the problem of the completeness of quantum theory on a physical level, and why a philosophical analysis was necessary. What Hermann particularly wanted to show is that quantum mechanics could still be seen as causal and complete, without having to assume some hidden causes."

► Further suggestion: Hermann could not / dared not be revolutionary? She stressed continuity – not radical changes. An adaptive attitude.

Final question: Why is her criticism still not known?

1. Her treatise is still not translated into English. (but this will change?)

2a. Max Jammer doesn't mention her in connection to Bell's criticism on von Neumann's argument.

2b. People who mention Grete Hermann's criticism usually only mention what Jammer wrote. But this will change from now on!

Contd: **Final question: Why is her criticism still not known?**

3a. Von Neumann's argument itself is not widely studied:

'A book more frequently referred to than read by physicists because of its mathematical sophistication.' (Redhead, 1987)

'Well, I suppose that they regard von Neumann's book as a perfectly adequate formal treatment for pedants, people who like that sort of thing [formal mathematics]. They wouldn't read it themselves but they're glad somebody has done all that hard work!' (Bastin, 1977)

Contd: **Final question: Why is her criticism still not known?**

3b. James Albertson (1961) in his accessible and therefore well-studied Dirac-formulation of the proof is not critical at all and relegates all assumptions, including the problematic one, to an appendix.

Remedy: read von Neumann's book!